# Riemannian Geometry
# Homework 1

Due on Wednesday, October 24, 2012

PROBLEM 1: (Stereographic Projection) Let $f : S^n - \{(0, \dots, 0, 1)\} \to \mathbb{R}^n$ be the stereographic projection from $N = (0, \dots, 0, 1)$. More precisely, $f$ sends a point $p$ on $S^n$ different from $N$ to the intersection $f(p)$ of the line $Np$ passing through $N$ and $p$ with the equatorial plane $x^{n+1} = 0$, as shown in the figure.

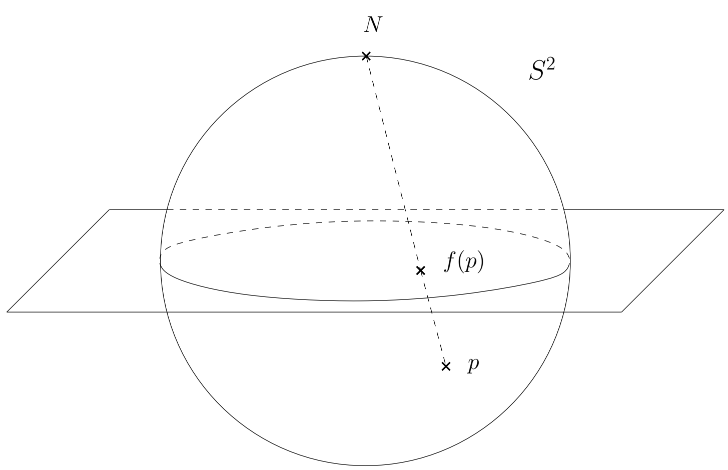


Figure 1: Stereographic Projection

a) Find an explicit formula for the stereographic projection map $f$.

b) Find an explicit formula for the inverse stereographic projection map $f^{-1}$.

c) If $S = -N$, $U = S^n - N$, $V = S^n - S$ and $g : S^n \to \mathbb{R}^n$ is the stereographic projection from $S$, then show that $(U, f)$ and $(V, g)$ form a $C^\infty$ atlas of $S^n$.

Problem 2: (The orthogonal group SO(3)) Let $M^{3\times3}$ denote the space of real 3 by 3 matrices. The orthogonal group SO(3) is then defined to be

$$\mathrm{SO}(3) = \{A \in M^{3\times3} | A^*A = \mathrm{Id} \text{ and } \det(A) = 1\}$$

a) Show that SO(3) is a smooth 3-dimensional submanifold of $M^{3\times3}$. Hint: use the implicit function theorem.

b) Show that the group $\mathrm{Sp}(1) = \{q \in \mathbb{H} | \bar{q}q = 1\}$ of unit quaternions is diffeomorphic to $S^3$.

c) Denote by $\mathfrak{sp}(1) = \{v \in \mathbb{H} | v + \bar{v} = 0\}$ the space of imaginary quaternions. Show that the action of Sp(1) on $\mathfrak{sp}(1)$ given by

$$q.v = qv\bar{q} \quad \text{for } q \in \mathrm{Sp}(1) \text{ and } v \in \mathfrak{sp}(1)$$

is smooth and well defined, i.e. it defines a smooth map from $\mathrm{Sp}(1) \times \mathfrak{sp}(1) \to \mathfrak{sp}(1)$. Deduce that there is a smooth group homomorphism from Sp(1) onto $SO(3)$. What is the kernel of this homomorphism? What is SO(3) thus diffeomorphic to?

Problem 3: (The tangent bundle $T(S^2)$ of $S^2$) We define the tangent bundle $T(S^2)$ of $S^2$ in the following way:

$$T(S^2) = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | \, ||p||^2 = 1 \text{ and } (p, v) = 0\}$$

where $(-, -)$ is an inner product on $\mathbb{R}^3$.

a) Show that $T(S^2)$ with the subspace topology inherited from $\mathbb{R}^3 \times \mathbb{R}^3$ is a topological 4-manifold.

b) Show that $T(S^2)$ admits a $C^\infty$ atlas such that the map $\pi : T(S^2) \to S^2$ which sends $(p, v)$ to $p$ is $C^\infty$.

c) The subset

$$T^1(S^2) = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | ||p||^2 = 1, \, (p, v) = 0 \text{ and } ||v||^2 = 1\}$$

is called the unit tangent bundle of $S^2$. Show that it is a smooth submanifold of $T(S^2)$ diffeomorphic to $\mathbb{R}P^3$.