Riemannian Geometry
Homework 10

Due on Friday, January 18, 2013

PROBLEM 1: (Second Bianchi Identity) Let D be a torsion-free linear connection on a manifold M and R its $(1,3)$-curvature tensor. Prove the *second Bianchi identity*:

$$D_x R(y,z) + D_y R(z,x) + D_z R(x,y) = 0.$$

Hint: Use the formula (definition of the covariant derivative of a $(1,3)$-tensor):

$$(D_X R)(Y,Z) = [D_X, R(Y,Z)] - R(D_X Y, Z) - R(Y, D_X Z)$$

for any vector fields X, Y, Z.

PROBLEM 2: (Ric < 0 and Killing fields) Let f be a smooth function on a pseudo-Riemannian manifold (M,g), and D the Levi-Civita connection of f. Recall that the gradient and the Hessian of f are defined by

$$df(v) = g(\operatorname{grad} f, v) \ \forall v, \quad \operatorname{Hess} f = Ddf,$$

i.e. $\operatorname{Hess} f(X,Y) = XYf - (D_X Y)f$ on vector fields.

(a) Show that $\operatorname{Hess} f(v,w) = g(D_v(\operatorname{grad} f), w)$.

(b) Let X be a *Killing vector field* (an infinitesimal isometry), i.e. DX is skew-adjoint, i.e. $g(D_V X, W) + g(D_W X, V) = 0$ for all vector fields V, W. Show that for $f = \frac{1}{2} g(X,X)$ we have

$$\operatorname{grad} f = -D_X X,$$

$$\operatorname{Hess} f(v,w) = g(D_v X, D_w X) - R(X,v,X,w).$$

(c) Show that on a compact Riemannian manifold with Ric < 0, every Killing vector field is identically 0. (Hint: At a maximum point of f, the Hessian of f is negative semi-definite).