Riemannian Geometry
Homework 11

Due on Friday, January 25, 2013

Problem 1: Let Γ be the subgroup of the 3-dimensional Heisenberg group Nil consisting of matrices with integer entries, i.e.

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x,\, y,\, z \in \mathbb{Z} \right\}$$

(see problem 3 in Homework 6). Let

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(i) Observe that $c = [a, b]$ and c is central in Γ (so $[a, c] = [b, c] = 1$). Conclude that any element of Γ has a unique "normal form" expression $a^\alpha b^\beta c^\gamma$, $\alpha,\, \beta,\, \gamma \in \mathbb{Z}$.

(ii) Consider the system of generators $\{a, b\}$ of Γ. Show that Γ has polynomial growth of order 4 with respect to $\{a, b\}$.

Hint: Were a, b, c generators, then the growth would be cubic. We have, however, $[a^m, b^l] = c^{ml}$ and so the length of c^n as the shortest word in generators a, b grows as $\sqrt{n}$.

(iii) Conclude that the compact manifold Nil/Γ considered in homework 6 does not admit a Riemannian metric with $Ric \geq 0$.