Riemannian Geometry
Homework 12

You do not have to hand it in!

PROBLEM 1: (Einstein manifolds) Let (M, g) be a Riemannian manifold, and D its Levi-Civita connection.

a) Let δ denote the divergence operator. For instance, the if h is a symmetric $(0, 2)$ form, then δh is a 1-form, defined by

$$\delta h = \mathrm{tr}_{12}(Dh),$$

where tr_{12} means that you take the trace over the first and second index. Using the second Bianchi identity (also known as the differential Bianchi identity, cf. homework 10, problem 1)), show that

$$\delta \mathrm{Ric} = \frac{1}{2} ds.$$

Also show that

$$\delta(\mathrm{Ric} - \frac{1}{2} g) = 0.$$

(The last fact is important in general relativity, where $G = \mathrm{Ric} - \frac{1}{2}g$ is known as the Einstein tensor.)

b) Assume that there is a smooth function λ such that

$$\mathrm{Ric} = \lambda g.$$

By using part a), show that if the dimension of M is at least 3, then λ must be constant.

Manifolds satisfying $\mathrm{Ric} = \lambda g$, where λ is a constant are called *Einstein manifolds*.