Riemannian Geometry
Homework 2

Due on Friday, November 2, 2012

PROBLEM 1: (Q18 of Chap.3 of Spivak, volume 1) Although there is no everywhere non-zero vector field on S^2, there is one on $S^2 - \{(0,0,1)\}$, which is diffeomorphic to $\mathbb{R}^2$, which looks like a magnetic dipole near the point $(0,0,1)$ (see the figure below):

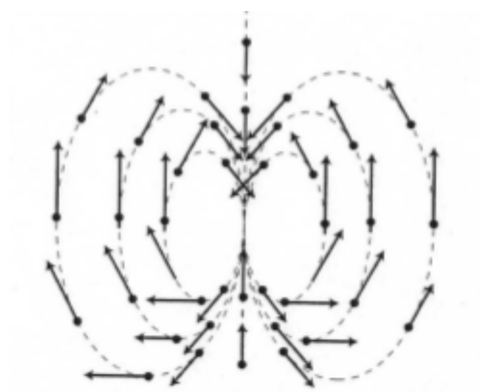

Figure 1: Magnetic Dipole (image taken from Spivak, volume 1)

PROBLEM 2: (Q19 of Chap.3 of Spivak, volume 1) Suppose we have a "multiplication" map $(a,b) \mapsto a.b$ from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ that makes $\mathbb{R}^n$ into a (non-associative) division algebra. That is,

$$\begin{aligned}(a_1 + a_2).b &= a_1.b + a_2.b \\ a.(b_1 + b_2) &= a.b_1 + a.b_2 \\ \lambda(a.b) &= (\lambda a).b = a.(\lambda b) \text{ for } \lambda \in \mathbb{R} \\ a.(1, 0, \ldots, 0) &= a\end{aligned}$$

and there are no zero-divisors:

$$a, b \neq 0 \implies a.b \neq 0.$$

(For example, for $n = 1$, we can use ordinary multiplication, and for $n = 2$, we can use "complex multiplication"). Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Under these assumptions (note that in fact, such a real non-associative division algebra was shown to exist only for $n =$ 1,2,4, or 8, this was proved independently by Michel Kervaire and John Milnor in 1958), show that:

a) Every point in S^{n-1} is $a.e_1$ for a unique $a \in \mathbb{R}^n$.

b) If $a \neq 0$, then $a.e_1, \ldots, a.e_n$ are linearly independent.

c) If $p = a.e_1 \in S^{n-1}$ then the projection of $a.e_2, \ldots, a.e_n$ on $T_p(S^{n-1})$ are linearly independent.

d) Multiplication by a is continuous.

e) $T(S^{n-1})$ is trivial.

f) $T(\mathbb{P}^{n-1})$ is trivial.

As a remark, for $n =$ 1,2, 4 and 8 respectively, there exists at least one such a non-associative division algebra, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively (real numbers, complex numbers, quaternions and octonions respectively).

PROBLEM 3: (Q29 of chap. 2 in Spivak, volume 1) Let $f : \mathbb{R}P^2 \to \mathbb{R}^3$ be the map

$$f([(x, y, z)]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy)$$

whose image is the Steiner surface, which is the subset of $\mathbb{R}^3$ defined by the equation

$$Y^2Z^2 + X^2Z^2 + X^2Y^2 = XYZ.$$

Show that f fails to be an immersion at 6 points.

Problem 4: (Immersions and submersions)

a) Show that an immersion between two n-dimensional manifolds is an open map.

b) Show that if $f : M \to N$ is a submersion, then the pull-back map $f^* : C^\infty(N, T^*(N)) \to C^\infty(M, T^*(M))$, from the space of 1-forms on N to the space of 1-forms on M is injective.