Riemannian Geometry
Homework 3

Due on Friday, November 9, 2012

PROBLEM 1: (Hyperbolic Space) Consider $\mathbb{R}^{n+1}$ with the Minkowski (indefinite) inner product:

$$< v, w >= -v^0 w^0 + v^1 w^1 + \ldots + v^n w^n.$$

Hyperbolic n-space can be defined as the subset

$$H^n = \{v \in \mathbb{R}^{n+1} | < v, v >= -1 \text{ and } v^0 > 0\}$$

with the induced metric g (g is called the hyperbolic metric) from the Minkowski inner product. Let τ denote pseudoinversion with pole

$$s = (-1, 0, \ldots, 0).$$

More precisely, τ is the map from the interior $I^+(s)$ of the future null cone of s to $I^+(s)$, which takes a point p to the point q, where q is the unique point on the future pointing ray from s passing through p, such that

$$||p - s||.||q - s|| = 1.$$

The norm used is the one coming from the Minkowski inner product.

a) Show that (the restriction of) τ is a diffeomorphism from H^n onto the open n-ball (of radius 1/2):

$$B^n = \{v \in \mathbb{R}^{n+1} | v^0 = -\frac{1}{2} \text{ and } < v, v >< 0\}.$$

b) Use the diffeomorphism constructed in part a) to transport the hyperbolic metric g from H^n onto B^n. In other words, if $f : B^n \to H^n$ is the inverse of the diffeomorphism constructed in a) (in fact f is the restriction of τ to B^n), then compute the pull-back metric $f^*(g)$ on B^n. Which model of hyperbolic space is this?

PROBLEM 2: (De Sitter and Anti de Sitter spaces) Equip $\mathbb{R}^{n+1}$ with the following quadratic forms:

$$< v, w >_1 = -v^0w^0 + v^1w^1 + \ldots + v^nw^n$$
$$< v, w >_2 = -v^0w^0 - v^1w^1 + v^2w^2 + \ldots + v^nw^n.$$

The de Sitter (respectively Anti de Sitter) space of dimension n is the submanifold of $\mathbb{R}^{n+1}$ defined by $< v, v >_1 = 1$ (respectively $< v, v >_2 = -1$) equipped with the induced pseudo-Riemannian metric. The de Sitter n-space is denoted by dS^n while anti de Sitter n-space is denoted by AdS^n.

a) Show that the induced metrics on dS^n and AdS^n are Lorentzian.

b) Show that the intersection with a spacelike hyperplane (i.e. a hyperplane of $\mathbb{R}^{n+1}$ such that each tangent vector v to the hyperplane satisfies $< v, v >> 0$) is isometric to S^{n-1} for dS^n and to (two copies of) H^n for AdS^n.

c) Show that dS^n is diffeomorphic to $S^{n-1} \times \mathbb{R}$, while AdS^n is diffeomorphic to $\mathbb{R}^{n-1} \times S^1$.

d) Using stereographic projection from the point $(-1, 0, \ldots, 0)$ onto the hyperplane $\{v^0 = 0\}$, show that Ads^4 is isometric to the following Riemannian manifold:

$$B = \{(x, y, z, u) \in \mathbb{R}^4 | x^2 + y^2 + z^2 - u^2 < 1\}$$

with Lorentzian metric

$$g = \frac{4}{(1 - s^2)^2}(dx^2 + dy^2 + dz^2 - du^2),$$

where $s = \sqrt{x^2 + y^2 + z^2 - u^2}$.