Riemannian Geometry
Homework 4

Due on Friday, November 16, 2012

PROBLEM 1: (Connections and Geodesics)

a) Prove that the torsion of a linear connection is a tensor, i.e. it is linear over $C^\infty(M)$ in both arguments.

b) Prove that the difference of two linear connections on a manifold is a tensor. Show that the two connections have the same parametrized geodesics if and only if their difference is an antisymmetric tensor.

c) Let g be a pseudo-Riemannian metric on M and $\tilde{g} = e^f g$, where $f \in C^\infty(M)$. Derive a relation between the Levi-Civita connections of g and $\tilde{g}$. Deduce that light-like (i.e. $g(\dot\gamma(t), \dot\gamma(t)) \equiv 0$) unparametrized geodesics on any pseudo-Riemannian manifold are conformally invariant.

d) Let D be a linear connection on a smooth manifold M. We can extend D to $(p, 0)$-tensors, i.e. sections of $\bigotimes^p T^*M$, by $S \mapsto D_X S$ where:

$$(D_X S)(X_1, \ldots, X_p) = L_X(S(X_1, \ldots, X_p)) - \sum_{i=1}^{p} S(X_1, \ldots, X_{i-1}, D_X X_i, \ldots, X_n).$$

Observe that

$$(X_0, X_1, \ldots, X_p) \mapsto (D_{X_0} S)(X_1, \ldots, X_p)$$

is a $(p+1, 0)$-tensor. Show that the Levi-Civita connection of a pseudo-Riemannian metric g satisfies

$$\nabla_X(g) = 0 \quad \text{for all } X \in \Gamma(TM).$$

Remark: one can extend D to act on (p,q)-tensors, p, $q \in \mathbb{N}$, i.e. on sections of $\bigotimes^p T^*M \otimes \bigotimes^q TM$.

e) Let ∇ be the Levi-Civita connection of a Riemannian metric g and $f \in C^\infty(M)$. Show that ∇df is a symmetric $(2,0)$-tensor (cf. part d)). ∇df is called the Riemannian Hessian of f. Give the expression of ∇df in a local chart.

Problem 2: (Surfaces of Revolution) Let M be a revolution surface in $\mathbb{R}^3$, endowed with the induced metric g.

a) The meridian line being parametrized by the length u, and the angle of rotation being denoted by θ, show that the metric is given by

$$g = du^2 + a^2(u)d\theta^2,$$

for some smooth function $a(u)$.

b) Show that the geodesics are:

1) the meridian lines,
2) the parallels ($u =$ constant) for which $a'(u) = 0$,
3) the curves which, when parametrized by length, satisfy

$$\left(\frac{du}{dt}^2\right) + a^2(u(t))\left(\frac{d\theta}{dt}^2\right) = 1,$$
$$a^2(u(t))\frac{d\theta}{dt} = C,$$

where C is a parameter associated to the geodesic.

c) Prove that the geodesics oscillate between two consecutive parallels satisfying $a(u) = C$, except in the case one of these parallels is extremal (that is $a'(u) = 0$): our geodesic is then asymptotic to this parallel, which is itself a geodesic.