Riemannian Geometry
Homework 5

Due on Friday, November 23, 2012

PROBLEM 1: (The Hyperbolic plane) Let (H^2, g) be the upper half-plane model of the hyperbolic plane, i.e.

$$H^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}$$
$$g = \frac{1}{y^2}(dx^2 + dy^2)$$

a) Show that the Christoffel symbols in coordinates $x^1 = x$, $x^2 = y$ are:

$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y}, \qquad \Gamma^2_{11} = \frac{1}{y},$$

and all other Γ^k_{ij} are 0.

b) Using the equation of geodesics in local coordinates, show that the geodesics in H^2 are (suitably parametrized) semi-circles with centre on the x-axis, together with the vertical rays parallel to the y-axis (i.e. $x = c$ for some constant $c \in \mathbb{R}$).

c) Show that these geodesics have infinite length in either direction.

d) Show that if γ is a geodesic in (H^2, g) and p is a point in H^2 not lying on γ (i.e. p does not belong to the image of γ), then there are infinitely many geodesics through p which do not intersect γ (i.e. their images do not intersect the image of γ).

PROBLEM 2: Let (M, g) be a Riemannian manifold, d the corresponding distance function.

a) Show that if c is the straight line joining v, $w \in T_m(M)$, then

$$\lim_{v,w \to 0} \frac{L(\exp \circ c)}{L(c)} = 1$$

(L denotes the length of a curve).

b) Similarly if $\gamma_{v,w}$ is the unique geodesic joining $\exp_m(v)$ and $\exp_m(w)$, and $c_{v,w} = \exp^{-1}(\gamma_{v,w})$, then

$$\lim_{v,w \to 0} \frac{L(\gamma_{v,w})}{L(c_{v,w})} = 1.$$

c) Conclude that

$$\lim_{v,w \to 0} \frac{d(\exp_m(v), \exp_m(w))}{||v - w||_g} = 1.$$

PROBLEM 3:

a) Let X, Y be (smooth) vector fields on a smooth manifold M, and Θ_t, Ψ_s the corresponding local flows. Show that $[X, Y] = 0$ if and only if $\Theta_t \circ \Psi_s = \Psi_s \circ \Theta_t$ for all t, s.

b) Let G be a Lie group with the Lie algebra $\mathfrak{g}$. Show that if v, $w \in \mathfrak{g}$ and $[v, w] = 0$, then

$$\exp(v) \exp(w) = \exp(v + w).$$

PROBLEM 4:

a) Let G be a Lie group of dimension n. Show that there exists a unique (up to a scalar factor) left invariant non-vanishing differential n-form $\Omega \in \Gamma(\Lambda^n T^* G)$. The corresponding measure is called the Haar measure on G.

b) Compute the left-invariant volume form on $\mathrm{GL}(n, \mathbb{R})$.

c) Let M be a smooth manifold and G a compact group acting smoothly on M. Show that there exists a G-invariant metric on M (so that G acts isometrically on M).

Optional Problem: (Poisson structure on $\mathfrak{g}^*$) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{g}^*$ the dual vector space. Identifying $(\mathfrak{g}^*)^*$ with $\mathfrak{g}$, we define an operation (Poisson bracket)

$$\{-,-\} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$$

by

$$\{f, g\}(x) = x([df|_x, dg|_x])$$

where the bracket is the Lie algebra bracket.

a) Show that $\{-,-\}$ makes $C^\infty(\mathfrak{g}^*)$ into a Lie algebra.

b) Show that $\{-,-\}$ satisfies the Leibniz rule:

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

c) Let $\mathfrak{g} = gl_n(\mathbb{R})$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via $A \mapsto L_A \in \mathfrak{g}^*$, where

$$L_A(v) = \mathrm{tr}(Av).$$

Show that the functions $\chi_k(A) = \mathrm{tr}(A^k)$ belong to the centre of the algebra $(C^\infty(\mathfrak{g}^*), \{-,-\})$, i.e.

$$\{\chi_k, f\} = 0, \qquad \text{for all } f \in C^\infty(\mathfrak{g}^*).$$