Riemannian Geometry
Homework 6

Due on Friday, November 30, 2012

PROBLEM 1: (The Fubini-Study metric) Compute the Fubini-Study metric on $\mathbb{C}P^n$ in local coordinates of your choice. Recall that this is the metric induced by the Riemannian submersion

$$S^{2n+1} \to S^{2n+1}/S^1 \simeq \mathbb{C}P^n,$$

where the action of $\mathrm{U}(1) \simeq S^1$ is

$$\lambda.(z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n),$$

where $\sum_{i=0}^n |z_i|^2 = 1$ and $|\lambda| = 1$ (the z_i and λ are of course complex numbers here).

PROBLEM 2: Let G consist of all elements of $\mathrm{GL}(3, \mathbb{R})$ of the form

$$\begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}$$

with $a > 0$. Prove that

a) G is a closed subgroup of $\mathrm{GL}(3, \mathbb{R})$,

b) G is non-abelian,

c) G does not admit a biinvariant pseudo-Riemannian metric.

Problem 3:

a) Compute the Killing form for the following Lie groups

  (i) $\mathrm{SO}(2,1)$

  (ii) the Heisenberg group (often denoted by Nil) of strictly upper-triangular real matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Give an example of a left-invariant metric on the Heisenberg group, i.e. a Nil-invariant metric on $\mathbb{R}^3$ (in coordinates x, y, z).

c) Let

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y, z \in \mathbb{Z} \right\}.$$

Show that Nil/Γ is a compact manifold.

Optional Problem: (optional problem for physicists) This is less of a problem, in the sense of having to solve it, than an example of a Lie-group interpretation of a mechanical system.

Consider a symmetric spinning top in $\mathbb{R}^3$, i.e. a rigid body with an axis of symmetry (through the origin), moving in vacuous $\mathbb{R}^3$ with one point fixed at the origin.

Write down the kinetic energy in terms of Euler's angles and principal moments of inertia. Observe that the kinetic energy defines a left-invariant metric on $SO(3)$ (in coordinates given by Euler's angles).

Let β be the Euler angle which ranges from 0 to π (the other two range from 0 to 2π). By allowing β to range from 0 to 2π, we obtain a left-invariant metric on the double cover of $SO(3)$, i.e. on $SU(2) \simeq S^3$. S^3 equipped with this metric is called the *Berger sphere*.