Riemannian Geometry
Homework 7

Due on Friday, December 7, 2012

PROBLEM 1: (The fixed point set of an isometry) Let (M, g) be a Riemannian manifold and $\varphi : M \to M$ an isometry. Let

$$N = \{m \in M; \varphi(m) = m\}.$$

a) Show that N is a smooth submanifold of M (Hint: consider $M \times M$).

b) Show that N is a *totally geodesic* submanifold of M, i.e. if $\gamma : I \to M$ is a geodesic such that $\gamma(t_0) \in N$ and $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}N$, for some $t_0 \in I$, then $\gamma(t) \in N$ for all $t \in I$.

PROBLEM 2: (The Hyperbolic plane revisited) Let

$$D = \{z \in \mathbb{C}; |z| < 1\}$$

be the unit disk equipped with the Poincaré metric

$$g(v, w) = \frac{4}{(1 - |z|^2)^2} v.w,$$

where $v.w$ is the usual scalar product of v and w in $\mathbb{R}^2$. We will sometimes use complex notation z for a point in D, or real notation (x, y) by decomposing $z = x + iy$ into real and imaginary parts.

a) Observe that the rotations $z \mapsto \lambda z$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, are isometries. Also observe that the reflection $\varphi(x, y) = (x, -y)$ is an isometry. Use problem 1 to show that straight lines through the origin O are geodesics in (D, g).

b) Let $\lambda \in (-1, 1)$ and
$$\psi(z) = \frac{z+\lambda}{1+\lambda z}.$$
Show that ψ is an isometry of (D, g), and the image of the line $(0, t)$ is a circle centered on the x-axis passing through $(\lambda, 0)$ and meeting the unit circle $\{x^2 + y^2 = 1\}$ orthogonally. Conclude that the geodesics in (D, g) are straight lines or circles which meet the boundary $\partial D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ orthogonally in both cases.

Problem 3: (Hopf map) Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the Riemannian submersion from the unit sphere S^{2n+1} onto $\mathbb{C}P^n$ with the Fubini-Study metric (the map π is known in the literature as the Hopf map). Let γ, $\delta : [0, \infty) \to \mathbb{C}P^n$ be two unit speed geodesics with $\gamma(0) = \delta(0)$ and $\gamma'(0) \neq \delta'(0)$. Determine the first time $T > 0$ such that $\gamma(T) = \delta(T)$ (Hint: there are two possibilities).

Optional Problem A: Let G be a connected Lie group with lie algebra $\mathfrak{g}$. For left-invariant vector fields X, Y on G, set
$$\nabla_X Y = \frac{1}{2}[X, Y].$$
Show that this determines a linear connection ∇ on G invariant under all left and right translations, as well as the map $j : g \mapsto g^{-1}$. Show that such a connection is unique.
<u>Remark:</u> A linear connection ∇ is said to be invariant under $f : M \to M$ if for any X, $Y \in \Gamma(M, TM)$,
$$\nabla_{df(X)} df(Y) = df(\nabla_X Y).$$

Optional Problem B: (Cosmological Redshift) Suppose that (B, g) and (F, h) are pseudo-Riemannian manifolds and $f > 0$ a smooth function on B. The *warped product* $M = B \times_f F$ is the product manifold $M = B \times F$ with the metric
$$g_M = g + f^2 h, \quad g_M|_{(x,y)}\big((v, w), (v, w)\big) = g(v, v) + f^2(x)h(w, w).$$

(a) Express a surface of revolution in $\mathbb{R}^3$ with the submanifold metric as a warped product.

(b) Show that $\pi : B \times_f F \to B$ is a Riemannian submersion and that the bracket of two horizontal vector fields is horizontal. Conclude that submanifolds of the form $B \times \{y\}$ are totally geodesic (see problem 1) in (M, g_M).

(c) Let $(B, g) = (I, -dt^2)$, where $I = (a, b) \in \mathbb{R}$ is a possibly infinite interval and (F, h) a Riemannian manifold, so that the metric on $M = I \times_f F$ is Lorentzian $g_M = -dt^2 + f^2 h$. We refer to such M as *Robertson-Walker spacetime* (not necessarily 4-dimensional).

(i) Let D be the Levi-Civita connection of g_M. Show that for vertical vector fields V, W on M, the horizontal component of $D_V W$ is $h(V, W) f f_t \frac{d}{dt}$. **Hint:** use the Koszul formula and the vanishing of torsion.

(ii) Show that a geodesic $\gamma(s) = (t(s), \alpha(s))$ in M satisfies

$$t'' + h(\alpha', \alpha') f(t) \frac{df}{dt} = 0$$

($'$ denotes derivative with respect to s).

(iii) Conclude that if $(t, \alpha(t))$ is a null-geodesic, then the function $f(t)t'$ is constant.

(iv) Consider now a photon (null geodesic with $t' > 0$) emitted at time t_p and observed at time $t_0 > t_p$ ($t = t(s)$ is the *galactic time*). The energy $E(s)$ of the photon measured by an observer at time $t(s)$ is t'. The frequency ν and the wavelength λ are related by $E = h\nu$, $\lambda\nu = c$, where h is the Planck constant and c the speed of light. Show that the *cosmological redshift* $z = \big(\lambda(t_0) - \lambda(t_p)\big)/\lambda(t_p)$ equals $\frac{f(t_0)}{f(t_p)} - 1$. (Thus the observed positive redshift implies that $f(t_0) > f(t_p)$, so the universe expands).