Riemannian Geometry
Homework 8

Due on Friday, December 21, 2012

PROBLEM 1: (O'Neill's formula) Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion and $\tilde{D}$, D the respective Levi-Civita connections. Recall that for horizontal lifts $\tilde{X}$, $\tilde{Y}$ of vector fields X, Y on M, the vector field $[\tilde{X}, \tilde{Y}] - [X, Y]$ is vertical, as is $[\tilde{X}, U]$, for any vertical vector field U. Moreover,

$$\tilde{D}_{\tilde{X}}\tilde{Y} = \widetilde{D_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^V.$$

a) Using the above, show that

$$\tilde{g}(\tilde{D}_U\tilde{X}, \tilde{Y}) = -\frac{1}{2}g([\tilde{X}, \tilde{Y}]^V, U).$$

b) Derive the O'Neill's formula for sectional curvatures $\tilde{K}$, K of $\tilde{g}$, g:

$$K(X, Y) = \tilde{K}(\tilde{X}, \tilde{Y}) + \frac{3}{4}\|[\tilde{X}, \tilde{Y}]^V\|^2.$$

c) Let now G/H be a homogeneous space with Riemannian metric induced by the biinvariant metric induced by the biinvariant metric $< -, - >$ on G. Write $\mathfrak{p}$ for $\mathfrak{h}^\perp$ and identify $\mathfrak{p}$ with $T_{[e]}G/H$. Show that, for X, $Y \in \mathfrak{p}$

$$K(X, Y) = \frac{1}{4}\|[X, Y]^\mathfrak{p}\|^2 + \|[X, Y]^\mathfrak{h}\|^2,$$

where $Z^\mathfrak{h}$, $Z^\mathfrak{p}$ denote components of Z in the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p},$$

(Note that this is not a Lie algebra decomposition in general, because $\mathfrak{p}$ need not be closed under $[-, -]$.)

Problem 2: (Sectional Curvature of $\mathbb{CP}^n$) Equip $U(n)$ with the biinvariant metric given by the trace form, i.e.

$$< X, Y >= -\frac{1}{2}\operatorname{tr}(XY), \qquad \text{for } X,\, Y \in \mathfrak{u}(n).$$

Consider the induced $U(n)$-invariant on $U(n+1)/(U(n)\times U(1))$, i.e. the Fubini-Study metric on $\mathbb{CP}^n$. In the notation of problem 1, c),

$$\mathfrak{p} = \left\{ \begin{pmatrix} & 0_n & & v_1 \\ & & & \vdots \\ & & & v_n \\ -\overline{v_1} & \dots & -\overline{v_n} & 0 \end{pmatrix} \right\}$$

Let $v = (v_1, \dots, v_n)^T$ be a unit vector in $\mathfrak{p}$ and w a unit vector in $\mathfrak{p}$ orthogonal to v. Then w can be written as

$$w = \cos(\alpha)w_0 + \sin(\alpha)(iv),$$

where w_0 is orthogonal to both v and iv. Using problem 1, c), show that

$$K(v, w) = 1 + 3\sin^2(\alpha),$$

and so the sectional curvature of $\mathbb{CP}^n$ varies between 1 and 4.