Riemannian Geometry
Homework 9

Due on Friday, January 11, 2013

Problem 1: (Curvature of hypersurfaces)

Let M be an oriented hypersurface in an oriented Riemannian manifold $(\tilde{M}, g)$. Let $\tilde{D}$ be the Levi-Civita connection of g and ν a unit normal vector field to M (i.e. $\nu_m \perp T_m M$ for all $m \in M$).

a) Let U, V be vector fields on $\tilde{M}$ which are tangent to M at any point of M. Show that the component $(\tilde{D}_U V)_m^\perp$ of $(\tilde{D}_U V)_m$ orthogonal to M depends only on the values $u = U_m$, $v = V_m$ of U, V at m and hence defines a $(0,2)$-tensor $l$on M by:

$$l(u,v)\nu_m = (\tilde{D}_U V)_m^\perp.$$

The tensor l is called the *second fundamental form* of M. Show that l is symmetric.

b) Show that the Riemannian curvature tensors $R, \tilde{R}$ of (M, g) and $(\tilde{M}, g)$ are related by

$$R(x,y,u,v) = \tilde{R}(x,y,u,v) + l(x,u)l(y,v) - l(x,v)l(y,u).$$

c) Let M be a hypersurface of the Euclidean space $\mathbb{R}^n$ defined by a submersion, i.e. $M = f^{-1}(0)$, where $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $\operatorname{grad} f \neq 0$ at any point of M. Set $\nu = \operatorname{grad} f / \|\operatorname{grad} f\|$ and show that

$$l(u,v) = \frac{\operatorname{Hess} f(u,v)}{\|\operatorname{grad} f\|}.$$

d) Show that the sectional curvature K of the two-sheeted hyperboloid

$$M = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 - z^2 = -1,\ \ z > 0\}$$

is given by $K(x, y, z) = (x^2 + y^2 + z^2)^{-2}$. Thus the sectional curvature is asymptotically zero. Show that the volume of M is infinite (hint: parametrise M as a surface of revolution).