A few simple facts about the hyperbolic plane

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In this very short article, we will consider the upper half-plane model of the hyperbolic plane, and show that the metric is complete, by explicitly writing down equations for the geodesics, and we will prove by an explicit computation that the sectional curvature (= the Gaussian curvature) is identically equal to -1.

Let

$$H = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\} \tag{1}$$

be the upper half-plane, with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{2}$$

This is the (conformal) *Poincare half-plane model* of the hyperbolic plane. Given an arbitrary metric

$$ds^2 = g_{ij}\, dx^i \otimes dx^j, \tag{3}$$

the Christoffel symbols of the associated Levi-Civita connection are given by:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l}). \tag{4}$$

Hence for the hyperbolic metric above, you get the following Christoffel symbols:

$$\Gamma^2_{11} = 1/y \tag{5}$$

$$\Gamma^2_{22} = -1/y \tag{6}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = -1/y. \tag{7}$$

Therefore the geodesic equations are

$$\ddot{x} y = 2\dot{x}\dot{y} \tag{8}$$

$$\ddot{y} y = \dot{y}^2 - \dot{x}^2, \tag{9}$$

where the dot is the (ordinary) derivative with respect to "time" (ie the parameter of the curve). As a reminder, the general geodesic equation is

$$\ddot{x}^k + \Gamma^k_{ij}\, \dot{x}^i \dot{x}^j = 0. \tag{10}$$

Now consider the curve

$$x = 0 \tag{11}$$
$$y = e^t. \tag{12}$$

One can check that it has unit velocity (with respect to the hyperbolic metric) and that it is a geodesic. How do you check that it is a geodesic? You just differentiate and check that the curve satisfies the geodesic equations that I wrote earlier.

Next, consider the curve

$$x = -r\tanh(t) \tag{13}$$
$$y = r\,\mathrm{sech}(t), \tag{14}$$

where r is a constant (the radius of the circle). Similarly, you check that it is a geodesic of unit speed.

Next, check that the geodesic equations are invariant under the transformation

$$x \to x + c \tag{15}$$
$$y \to y \tag{16}$$

(ie horizontal translations). Combine this fact with the previous 2 types of geodesics, and you will get a lot of geodesics. In fact, up to affine reparametrization (which changes the initial position and the velocity of the geodesic), these are *all* the geodesics (if you also include the constant geodesics and allow for a change of direction of the geodesic). Indeed, consider an arbitrary point in H, and a tangent vector at that point. If the vector is 0, the geodesic with such initial conditions is the constant one. If the vector is non-zero and vertical, then there is a unique geodesic of the first type passing through it, and if not, then there is a unique circle with center on the x-axis which passes through the point and is tangent to the vector. In other words, in that last case, there is a unique geodesic of the second type which passes through it. Hence, we have shown (up to a few details) that the hyperbolic plane is complete (since the geodesics are defined for all time).

Now for the curvature. Using the Christoffel symbols computed above, and the formula

$$R^l_{ijk} = \Gamma^s_{ik}\Gamma^l_{js} - \Gamma^s_{jk}\Gamma^l_{is} + \frac{\partial}{\partial x^j}\left(\Gamma^l_{ik}\right) - \frac{\partial}{\partial x^i}\left(\Gamma^l_{jk}\right), \tag{17}$$

we get

$$R^2_{121} = -1/y^2. \tag{18}$$

Hence

$$R_{1212} = g_{22}\,R^2_{121} = -1/y^4. \tag{19}$$

We conclude that the Gaussian curvature is

$$K = R_{1212}/(g_{11}g_{22} - g_{12}g_{21}) \equiv -1, \tag{20}$$

as claimed.