Riemannian Geometry
Homework 1 and Solution

Due on Wednesday, October 24, 2012

PROBLEM 1: (Stereographic Projection) Let $f : S^n - \{(0,\ldots,0,1)\} \to \mathbb{R}^n$ be the stereographic projection from $N = (0,\ldots,0,1)$. More precisely, f sends a point p on S^n different from N to the intersection $f(p)$ of the line Np passing through N and p with the equatorial plane $x^{n+1} = 0$, as shown in the figure.

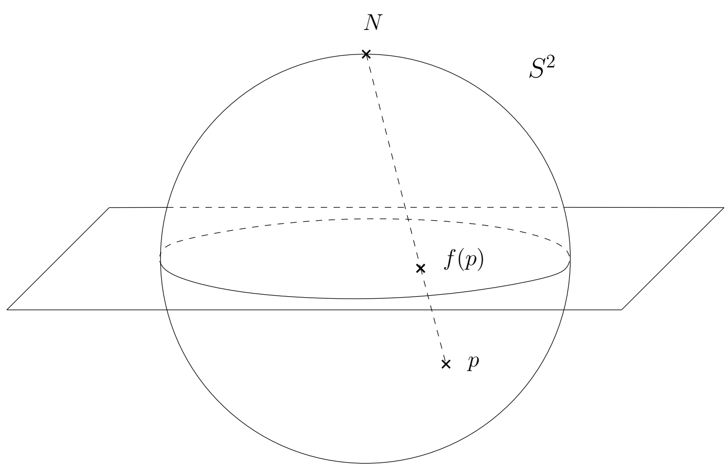

Figure 1: Stereographic Projection

a) Find an explicit formula for the stereographic projection map f.

Stereographic projection $f : S^n - \{(0,\ldots,0,1)\} \to \mathbb{R}^n$ *is given by:*

$$f(x^1,\ldots,x^{n+1}) = \frac{1}{1-x^{n+1}}(x^1,\ldots,x^n).$$

b) Find an explicit formula for the inverse stereographic projection map f^{-1}.

The inverse stereographic projection f^{-1} is given by:

$$f^{-1}(y^1, \ldots, y^n) = \frac{1}{||y||^2 + 1}(2y^1, \ldots, 2y^n, ||y||^2 - 1).$$

Here $||y||^2 = \sum_{i=1}^n (y^i)^2$. One can then easily check by direct computation that f and the map above are left and right inverses of each other.

c) If $S = -N$, $U = S^n - N$, $V = S^n - S$ and $g : S^n \to \mathbb{R}^n$ is the stereographic projection from S, then show that (U, f) and (V, g) form a C^∞ atlas of S^n.

The transition map $g \circ f^{-1} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ is given by:

$$g \circ f^{-1}(y^1, \ldots, y^n) = \frac{1}{||y||^2}(y^1, \ldots, y^n),$$

which is smooth, and moreover has a smooth inverse, given by the same formula. The geometric explanation for this is that $g \circ f^{-1}$ represents inversion with respect to the sphere $S^{n-1} \subset \mathbb{R}^n - \{0\}$, which is an involution (an involution τ of a set X is a map from X to X such that $\tau^2 = \mathrm{Id}$). Hence S^n has a smooth atlas consisting of just two patches (U, f) and (V, g). Moreover, one can show that this atlas is compatible with the atlas we constructed in tutorial, so that these two atlases define the same smooth structure on S^n.

Problem 2: (The orthogonal group SO(3)) Let $M^{3\times3}$ denote the space of real 3 by 3 matrices. The orthogonal group SO(3) is then defined to be

$$\mathrm{SO}(3) = \{A \in M^{3\times3} | A^*A = \mathrm{Id} \text{ and } \det(A) = 1\}.$$

a) Show that SO(3) is a smooth 3-dimensional submanifold of $M^{3\times3}$. Hint: use the implicit function theorem.

Let $f : M^{3\times 3} \to M^{3\times 3}$ *be given by*

$$f(A) = A^*A - \mathrm{Id}\,.$$

Let $A(t) : (-\epsilon, \epsilon) \to M^{3\times 3}$ *be a smooth curve with* $A(0) = A$ *and* $A'(0) = \dot{A}$. *Then*

$$df_A(\dot{A}) = \frac{d}{dt}(f(A(t)))|_{t=0} = \dot{A}^*A + A^*\dot{A}.$$

Fix some $G \in \mathrm{SO}(3)$. *Then* $df_G : M^{3\times 3} \to M^{3\times 3}$ *is* $\mathbb{R}$*-linear, and its kernel is*

$$\begin{aligned} \ker(df_G) &= \{\dot{G} \in M^{3\times 3} | G^*\dot{G} = X \textit{ for some skew-adjoint } X \in M^{3\times 3}\} \\ &= \{\dot{G} \in M^{3\times 3} | \dot{G} = GX \textit{ for some skew-adjoint } X \in M^{3\times 3}\} \end{aligned}$$

(Reminder: $X \in M^{3\times 3}$ *is said to be* skew-adjoint *if* $X + X^* = 0$.*) This shows that* $\ker(df_G)$ *is* 3*-dimensional, and naturally isomorphic to the space of skew-adjoint matrices in* $M^{3\times 3}$*, which we will denote by* $\mathfrak{so}(3)$. *Hence* f *has constant rank equal to* $3\times 3 - 3 = 6$ *at any point of* $\mathrm{SO}(3)$. *More generally, a similar proof shows that* f *has constant rank equal to* 6 *on* $\mathrm{GL}(3, \mathbb{R})$*, which is an open neighborhood of* $\mathrm{SO}(3)$ *in* $M^{3\times 3}$. *So by the constant rank theorem,* $\mathrm{SO}(3)$ *is a smooth* 3*-submanifold of* $\mathrm{GL}(3, \mathbb{R})$ *(itself an open submanifold of* $M^{3\times 3}$*).*

We remark that the condition $\det(A) = 1$ *does not affect the proof above, because by taking determinant of the equation* $A^*A = \mathrm{Id}$*, we get that* $\det(A)^2 = 1$*, so that* $\det(A) = +1$ *or* -1*, and* $\mathrm{SO}(3)$ *is actually the connected component of* $O(3)$ *containing the identity.*

b) Show that the group $\mathrm{Sp}(1) = \{q \in \mathbb{H} | \bar{q}q = 1\}$ of unit quaternions is diffeomorphic to S^3.

An arbitrary quaternion q *can be written uniquely as*

$$q = a + ib + jc + kd \textit{ for } a,\, b,\, c \textit{ and } d \textit{ real numbers.}$$

The condition $\bar{q}q = 1$ *then amounts to*

$$a^2 + b^2 + c^2 + d^2 = 1,$$

i.e. $(a, b, c, d) \in S^3$. *The map from* $\mathbb{R}^4 \to \mathbb{H}$ *which maps* (a, b, c, d) *to* $a + ib + jc + kd$ *is a diffeomorphism which thus restricts to a diffeomorhism from* S^3 *onto* $\mathrm{Sp}(1)$.

c) Denote by $\mathfrak{sp}(1) = \{v \in \mathbb{H} | v + \bar{v} = 0\}$ the space of imaginary quaternions. Show that the action of Sp(1) on $\mathfrak{sp}(1)$ given by

$$q.v = qv\bar{q} \quad \text{for } q \in \mathrm{Sp}(1) \text{ and } v \in \mathfrak{sp}(1)$$

is smooth and well defined, i.e. it defines a smooth map from $\mathrm{Sp}(1) \times \mathfrak{sp}(1)$ to $\mathfrak{sp}(1)$. Deduce that there is a smooth group homomorphism from Sp(1) onto $SO(3)$. What is the kernel of this homomorphism? What is SO(3) thus diffeomorphic to?

We first show that $qv\bar{q} \in \mathfrak{sp}(1)$ if $q \in Sp(1)$ and $v \in \mathfrak{sp}(1)$. Indeed

$$qv\bar{q} + \overline{qv\bar{q}} = qv\bar{q} + q\bar{v}\bar{q} = 0,$$

since $v + \bar{v} = 0$. Thus the action is well defined. The map it defines is in fact a homogeneous polynomial map in (a, b, c, d) and (v^1, v^2, v^3) (where $v = v^1 i + v^2 j + v^3 k$) quadratic in (a, b, c, d) and linear in (v^1, v^2, v^3), and is therefore smooth. We claim that the action defines a group homomorphism from $\mathrm{Sp}(1)$ into $\mathrm{GL}(\mathfrak{sp}(1)) \simeq \mathrm{GL}(3, \mathbb{R})$. If q_1 and $q_2 \in \mathrm{Sp}(1)$, then

$$(q_1 q_2).v = (q_1 q_2) v \overline{(q_1 q_2)} = q_1 q_2 v \bar{q}_2 \bar{q}_1 = q_1.(q_2 v).$$

Moreover, if $1.v = v$ for any $v \in \mathfrak{sp}(3)$. Also $q.(v + w) = q.v + q.w$ and $q.(cv) = cq.v$, for any $c \in \mathbb{R}$, $q \in \mathrm{Sp}(1)$ and $v, w \in \mathfrak{sp}(3)$. This shows that the action induces a group homomorphism $\rho : \mathrm{Sp}(1) \to \mathrm{GL}(\mathfrak{sp}(1))$ (ρ is clearly smooth). We want to show that ρ actually maps $\mathrm{Sp}(1)$ to $\mathrm{SO}(\mathfrak{sp}(1))$. A natural inner product on $\mathfrak{sp}(1)$ is given by

$$< v, w >= \mathfrak{Re}(\bar{v}w).$$

By the polarization formula, to show that a linear map L preserves an inner product, it suffices to show that $< Lv, Lv >=< v, v >$ for any vector v. We now compute:

$$< q.v, q.v >= (q\bar{v}\bar{q})(qv\bar{q}) = q\bar{v}v\bar{q} = \bar{v}v =< v, v > .$$

This shows that $\rho : \mathrm{Sp}(1) \to O(\mathfrak{sp}(1))$. But $\mathrm{Sp}(1)$ is connected, so its image under ρ (which is continuous) is contained in the connected component of $O(\mathfrak{sp}(1))$, namely $\mathrm{SO}(\mathfrak{sp}(1))$.

By possibly chosing a different orthonormal frame of $\mathfrak{sp}(1)$, we can always make sure that a given element $g \in \mathrm{SO}(\mathfrak{sp}(1))$ is of the form:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix},$$

with $a^2 + b^2 = 1$. Let $z \in \mathbb{C} \subset \mathbb{H}$ be a solution to

$$z^2 = a + bi.$$

Then

$$z.j = zj\bar{z} = z^2 j = (a + bi)j = ja + kb,$$
$$z.k = zk\bar{z} = z^2 k = (a + bi)k = -jb + ka.$$

Hence $\rho : \mathrm{Sp}(1) \to \mathrm{SO}(\mathfrak{sp}(1))$ is onto. Its kernel is easily seen to be the subgroup $\{+\mathrm{Id}, -\mathrm{Id}\}$. Hence $\mathrm{SO}(3)$ is diffeomorphic to S^3 modulo the antipodal relation, from which we conclude that $\mathrm{SO}(3)$ is diffeomorphic to real projective 3-space $\mathbb{R}P^3$.

PROBLEM 3: (The tangent bundle $T(S^2)$ of S^2) We define the tangent bundle $T(S^2)$ of S^2 in the following way:

$$T(S^2) = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | \, ||p||^2 = 1 \text{ and } (p, v) = 0\}$$

where $(-, -)$ is an inner product on $\mathbb{R}^3$.

a) Show that $T(S^2)$ with the subspace topology inherited from $\mathbb{R}^3 \times \mathbb{R}^3$ is a topological 4-manifold.

We only need to show that $T(S^2)$ is locally Euclidean, because it is a subspace of $\mathbb{R}^3 \times \mathbb{R}^3$ with the subspace topology, and we know that the latter is Hausdorff and second countable. Let $U_i^+ = D \times \mathbb{R}^2$ (similarly $U_i^- = D \times \mathbb{R}^2$), $1 \leq i \leq 3$, where D is the open unit disk in R^2. Each U_i^+ is clearly an open subset of $\mathbb{R}^4$. We define $f_i^+ : U_i^+ \to T(S^2)$ ($f_i^- : U_i^- \to T(S^2)$) by:

$$f_i^+(x, y, u, v) = ((x, \ldots, \sqrt{1 - x^2 - y^2}, \ldots, y), (u, \ldots, \frac{-ux - vy}{\sqrt{1 - x^2 - y^2}}, \ldots, v)),$$

where the square root term is at the i'th position. Define $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$ *by*

$$g_i((\mathbf{x}, \mathbf{u})) = ((x^1, \ldots, \widehat{x^i}, \ldots, x^3), (u^1, \ldots, \widehat{u^i}, \ldots, u^3)).$$

By covering $T(S^2)$ *with corresponding open sets* V_i^+ *and* V_i^-*, it is clear that* f_i^+ *is a homeomorphism from* U_i^+ *onto* V_i^+ *with inverse* g_i *restricted to* V_i^+ *(and similarly for* f_i^- *and the restriction of* g_i *to* V_i^-*).*

b) Show that $T(S^2)$ admits a C^∞ atlas such that the map $\pi : T(S^2) \to S^2$ which sends (p, v) to p is C^∞.

We claim that the homeomorphisms f_i^+, f_i^- *and the corresponding open covering of* $T(S^2)$ *by* V_i^+ *and* V_i^- *form a smooth atlas. Assume that*

$$x^2 + y^2 < 1$$
$$y > 0.$$

We compute for instance $(f_2^+)^{-1} \circ (f_3^+)$*:*

$$\begin{aligned} &(f_2^+)^{-1} \circ (f_3^+)((x, y), (u, v)) \\ =&((x, \sqrt{1 - x^2 - y^2}), (u, \frac{-ux - vy}{\sqrt{1 - x^2 - y^2}})), \end{aligned}$$

which is smooth and has a smooth inverse (one can check that its inverse is itself, i.e. that it is an involution). The projection π *is* C^∞ *as can be seen in a local chart:*

$$\pi((x, y), (u, v)) = (x, y).$$

c) The subset

$$T^1(S^2) = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | ||p||^2 = 1,\ (p, v) = 0 \text{ and } ||v||^2 = 1\}$$

is called the unit tangent bundle of S^2. Show that it is a smooth submanifold of $T(S^2)$ diffeomorphic to $\mathbb{R}P^3$.

The map $f : T(S^2) - (S^2 \times 0) \to \mathbb{R}$ *sending* (p, v) *to* $||v||^2 - 1$ *is smooth and has constant rank equal to* 1 *(since we removed the subset having* $v = 0$*). Hence* $f^{-1}(0)$ *is either empty or is a smooth submanifold of*

$T(S^2) - (S^2 \times 0)$ *(itself an open submanifold of* $T(S^2)$*) of dimension equal to* $\dim(T(S^2) - (S^2 \times 0))$ *minus* $rank(f)$*, i.e.* 3*. It is easy to see that* $f^{-1}(0)$ *is nonempty.*

We will then show that there is a diffeomorphism from $\mathrm{SO}(3)$ *onto* $T^1(S^2)$*, which will finish the proof, since we have already proved that* $\mathrm{SO}(3)$ *is diffeomorphic to* $\mathbb{R}P^3$*. Write a matrix* g *in* $\mathrm{SO}(3)$ *as*

$$g = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix},$$

where v_i *is the* i*'th column of* g*. Map* g *to* $(v_1, v_2) \in T^1(S^2)$*. The map is easily seen to be a diffeomorphism. In fact, its inverse simply maps* $(p, v) \in T^1(S^2)$ *to* $(p, v, p \times v) \in \mathrm{SO}(3)$*.*