Solution to Homework 10:

(1) $R(Y,Z)W = -D_Y D_Z W + D_Z D_Y W + D_{[Y,Z]} W$

$$(D_X R)(Y,Z)W = D_X\left(R(Y,Z)W\right) - R(D_X Y, Z)W - R(Y, D_X Z)W - R(Y,Z)D_X W$$

$$\Rightarrow (D_X R)(Y,Z)W = -D_X D_Y D_Z W + D_X D_Z D_Y W + D_X D_{[Y,Z]} W$$
$$+ D_{D_X Y} D_Z W - D_Z D_{D_X Y} W - D_{[D_X Y, Z]} W$$
$$+ D_Y D_{D_X Z} W - D_{D_X Z} D_Y W - D_{[Y, D_X Z]} W$$
$$+ D_Y D_Z D_X W - D_Z D_Y D_X W - D_{[Y,Z]} D_X W$$

$\Rightarrow (D_X R)(Y,Z)W$ + cyclic

$$= -D_{[D_X Y, Z]} W - D_{[D_Y Z, X]} W - D_{[D_Z X, Y]} W$$
$$- D_{[Y, D_X Z]} W - D_{[Z, D_Y X]} W - D_{[X, D_Z Y]} W$$
$$= -D_{[[X,Y],Z]} W - D_{[[Y,Z],X]} W - D_{[[Z,X],Y]} W \quad (\text{since } T^D = 0)$$
$$= 0 \quad \text{by Jacobi identity}$$

(2) a) $g(D_v Df, w)$

$= g(D(D_v f), w)$ since $T^D \equiv 0$

$= D_w D_v f$

$= D_v D_w f$

$= (\text{Hess } f)(v, w)$

b) Let $X \to$ Killing field

$f = \frac{1}{2} g(X, X)$

$\Rightarrow df(Y) = g(\nabla_Y X, X)$ (since ∇ is metric)

$= - g(\nabla_X X, Y)$ (by the Killing equation)

for any (C^∞) vector field Y

$\Rightarrow \boxed{\nabla f = - \nabla_X X}$

$(\text{Hess } f)(v, w)$

$\cancel{= - X g(D_v X, w) + g(D_v X, D_X}$

$= - g(D_v D_X X, w)$

$= - g(D_X D_v X, w) + R(v, X, X, w) - g(D_{[v,X]} X, w)$

$= - X g(D_v X, w) + g(D_v X, D_X w) - R(X, v, X, w) - g(D_{[v,X]} X, w)$

$= - X g(D_v X, w) + g(D_v X, D_X w) - R(X, v, X, w) + g(D_w X, [v, X])$

$= \underbrace{g(D_v X, D_w X) - R(X, v, X, w)}_{\text{symmetric in } v \,\&\, w} \underbrace{- X g(D_v X, w) + g(D_v X, [X, w]) - g(D_w X, [X, v])}_{\text{skew-symmetric in } v \,\&\, w}$

But ∇ is torsion-free, so $(\mathrm{Hess}\, f)(v,w) = (\mathrm{Hess}\, f)(w,v)$

$$\Rightarrow (\mathrm{Hess}\, f)(v,w) = \tfrac{1}{2}(\mathrm{Hess}\, f)(v,w) + \tfrac{1}{2}(\mathrm{Hess}\, f)(w,v)$$

$$= g(D_v X, D_w X) - R(X, v, X, w)$$

(c) <u>Proof by contradiction:</u> (Bochner method)

Assume (M,g) compact, $\mathrm{Ric} < 0$ and X is a non-zero Killing field.

Let p_{max} be a point at which f is maximal (there exists such a point by compactness).

$$\Rightarrow (\mathrm{Hess}\, f)_{p_{max}} \leq 0$$

Let $e_1, \dots, e_n$ be an orthonormal frame at p_{max}. Then:

$$\sum_{i=1}^{n} (\mathrm{Hess}\, f)(e_i, e_i) = \sum_{i=1}^{n} \| D_{e_i} X \|^2 - \mathrm{Ric}(X,X) \quad (\text{at } p_{max})$$

(by the formula for $(\mathrm{Hess}\, f)(v,w)$ above).

Since X non-zero, $f(p_{max}) > 0$ and $X_{p_{max}} \neq 0$, so

$$-\mathrm{Ric}(X_{p_{max}}, X_{p_{max}}) > 0$$

$$\Rightarrow \sum_{i=1}^{n} (\mathrm{Hess}\, f)_{p_{max}}(e_i, e_i) \geq -\mathrm{Ric}(X_{p_{max}}, X_{p_{max}}) > 0 \quad (\text{at } p_{max})$$

contradicting that $(\mathrm{Hess}\, f)_{p_{max}} \leq 0$. QED

Or $\left\{ \begin{array}{l} \mathrm{tr\,Hess} \leq 0,\ 0 \geq \mathrm{tr\,Hess} = \cancel{\text{positive}} \geq 0 - \mathrm{Ric}(X,X) \\ \Rightarrow \mathrm{Ric}(X,X) = 0 \\ \Rightarrow X = 0 \Rightarrow f = 0 \text{ at maximum} \end{array} \right\}$