Solution to Homework 11:

① (i) By calculation, we get

$$a^\alpha b^\beta c^\gamma = \begin{pmatrix} 1 & \alpha & \gamma+\alpha\beta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

Choose $\alpha = x$, $\beta = y$, $\gamma = z - xy$

Then $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = a^x b^y c^{z-xy}$

Assume $a^{\alpha_1} b^{\beta_1} c^{\gamma_1} = a^{\alpha_2} b^{\beta_2} c^{\gamma_2}$

$$\Rightarrow \begin{pmatrix} 1 & \alpha_1 & \gamma_1+\alpha_1\beta_1 \\ 0 & 1 & \beta_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_2 & \gamma_2+\alpha_2\beta_2 \\ 0 & 1 & \beta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1 = \alpha_2 \\ \beta_1 = \beta_2 \\ \gamma_1 + \alpha_1\beta_1 = \gamma_2 + \alpha_2\beta_2 \end{cases}$$

So also $\gamma_1 = \gamma_2$, and we get both existence and uniqueness of the normal form expression of an element of Γ as $a^\alpha b^\beta c^\gamma$, where $\alpha, \beta, \gamma \in \mathbb{Z}$.

(ii) Every $g \in \Gamma$ can be written uniquely as:

$g = a^\alpha b^\beta c^\gamma$ for some unique $\alpha, \beta, \gamma \in \mathbb{Z}$.

$l_{\{a,b\}}(c^\gamma) = O(\sqrt{|\gamma|})$ as $\gamma \to \infty$,

where $l_{\{a,b\}}(g)$ is the length of g with respect to the set of generators $\{a, b\}$ [and their inverses).

Indeed $[a^m, b^l] = c^{ml} \Rightarrow l_{\{a,b\}}(c^\gamma) = O(\sqrt{|\gamma|})$, as $\gamma \to \infty$

$\Rightarrow l_{\{a,b\}}(a^\alpha b^\beta c^\gamma)$ is $O(|\alpha| + |\beta| + \sqrt{|\gamma|})$

$\Rightarrow$ the growth of Γ with respect to $\{a, b\}$ is of order 4 (because

$$\left|\{(\alpha, \beta, \gamma) \in \mathbb{Z}^3 \mid |\alpha| + |\beta| + \sqrt{|\gamma|} \leq n\}\right| = O(n^4) \quad).$$

(iii) (Milnor) If (M^n, g) is compact with $Ric \geq 0$, then growth of $\pi_1(M^n)$ is of order $\leq n$, where $n = \dim M$.

But in this case, $n = 3$ and the manifold is compact and has $\pi_1(M) = \Gamma$ of growth of order $4 > n$, so that Nil/Γ does not admit a metric with $Ric \geq 0$.