Riemannian Geometry
Homework 2 and Solution

Due on Friday, November 2, 2012

PROBLEM 1: (Q18 of Chap.3 of Spivak, volume 1) Although there is no everywhere non-zero vector field on S^2, there is one on $S^2 - \{(0,0,1)\}$, which is diffeomorphic to $\mathbb{R}^2$, which looks like a magnetic dipole near the point $(0,0,1)$ (see the figure below):

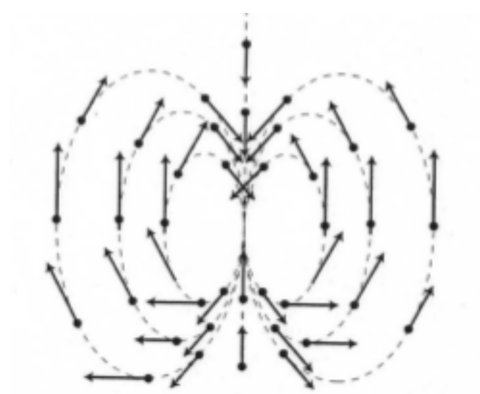

Figure 1: Magnetic Dipole (image taken from Spivak, volume 1)

We will make use of complex coordinates, because S^2 can easily be seen to admit a complex curve structure, i.e. it is a complex manifold of dimension 1, by using stereographic projection. If N is the North pole, and S is the South pole, then $S^2 - \{N\}$ has a complex coordinate z, while $S^2 - \{S\}$ has a complex coordinate w, with holomorphic transition

$$w = 1/z, \quad \text{for } z \neq 0.$$

N corresponds to $w = 0$. Looking at the figure, we define X on $S^2 - \{S\}$ by

$$X = iw^2 \partial_w - i\bar{w}^2 \partial_{\bar{w}^2}.$$

However, we also want X to extend smoothly to $z = 0$ corresponding to the point S, and we want X not to vanish on $S^2 - \{N\}$. In terms of the coordinate z, X is given by

$$X = i(\partial_{\bar{z}} - \partial_z),$$

which indeed is a smooth real vector field defined on all of S^2 (including at $z = 0$ corresponding to the point S), vanishing only at N, and which looks like the figure above near N.

PROBLEM 2: (Q19 of Chap.3 of Spivak, volume 1) Suppose we have a "multiplication" map $(a, b) \mapsto a.b$ from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ that makes $\mathbb{R}^n$ into a (non-associative) division algebra. That is,

$$\begin{aligned}
(a_1 + a_2).b &= a_1.b + a_2.b \\
a.(b_1 + b_2) &= a.b_1 + a.b_2 \\
\lambda(a.b) &= (\lambda a).b = a.(\lambda b) \text{ for } \lambda \in \mathbb{R} \\
a.(1, 0, \ldots, 0) &= a
\end{aligned}$$

and there are no zero-divisors:

$$a, b \neq 0 \implies a.b \neq 0.$$

(For example, for $n = 1$, we can use ordinary multiplication, and for $n = 2$, we can use "complex multiplication"). Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Under these assumptions (note that in fact, such a real non-associative division algebra was shown to exist only for n = 1,2,4, or 8, this was proved independently by Michel Kervaire and John Milnor in 1958), show that:

a) Every point in S^{n-1} is $a.e_1$ for a unique $a \in \mathbb{R}^n$.

   Let $p \in S^{n-1}$, then $p.e_1 = p$, which shows existence. For uniqueness, assume that $a.e_1 = p$. Then $a.e_1 = a$, showing that $a = p$, showing uniqueness.

b) If $a \neq 0$, then $a.e_1, \ldots, a.e_n$ are linearly independent.

Assume there exist real numbers $c_1,\ldots,c_n$ *such that*

$$c_1 a.e_1 + \ldots c_n a.e_n = 0.$$

We can rewrite this as

$$a.(c_1 e_1 + \ldots + c_n e_n) = 0.$$

But $a \neq 0$ *and there are no non-zero divisors, so we must have*

$$c_1 e_1 + \ldots + c_n e_n = 0.$$

Hence all the c_i *must vanish, proving the claim.*

c) If $p = a.e_1 \in S^{n-1}$ then the projection of $a.e_2, \ldots, a.e_n$ on $T_p(S^{n-1})$ are linearly independent.

Define

$$X_i = a.e_i - (a.e_i, a.e_1)a.e_1,$$

for $2 \leq i \leq n$. *We would like to show that the* X_i *are linearly independent. Assume there are real numbers* $c_2, \ldots, c_n$ *such that*

$$c_2 X_2 + \ldots c_n X_n = 0.$$

Then, by part b), it follows that $c_2,\ldots,c_n$ *must all vanish.*

d) Multiplication by a is continuous.

Multiplication by a *is a linear map from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$*, and any linear endomorphism of a* finite-dimensional *vector space is continuous.*

e) $T(S^{n-1})$ is trivial.

We have seen that $X_2, \ldots, X_n$ *are continuous vector fields on* S^{n-1} *which are linearly independent at each point* $p \in S^{n-1}$*. In fact, with respect to the standard smooth structure on* S^{n-1}*, they are even smooth (since left multiplication by* a *is smooth, and the projection map is smooth). This implies that the tangent bundle* $T(S^{n-1})$ *of* S^{n-1} *is trivial.*

f) $T(\mathbb{P}^{n-1})$ is trivial.

> *I claim that the smooth vector fields $X_2, \ldots, X_n$ descend do the quotient. In other words, there are smooth vector fields $\overline{X}_2, \ldots, \overline{X}_n$ such that*
>
> $$p_*(X_i) = \overline{X}_i, \text{ for } 2 \le i \le n,$$
>
> *where $p : S^{n-1} \to \mathbb{R}P^{n-1}$ is the standard smooth double covering map. Indeed, this is so because $\tau_*(X_i) = X_i$, for $2 \le i \le n$, where τ is the antipodal map.*

As a remark, for $n = 1$,2, 4 and 8 respectively, there exists at least one such a non-associative division algebra, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively (real numbers, complex numbers, quaternions and octonions respectively).

PROBLEM 3: (Q29 of chap. 2 in Spivak, volume 1) Let $f : \mathbb{R}P^2 \to \mathbb{R}^3$ be the map

$$f([(x, y, z)]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy)$$

whose image is the Steiner surface, which is the subset of $\mathbb{R}^3$ defined by the equation

$$Y^2Z^2 + X^2Z^2 + X^2Y^2 = XYZ.$$

Show that f fails to be an immersion at 6 points.

We define the map $F : S^2 \to \mathbb{R}^3$ by

$$F(x, y, z) = (yz, xz, xy).$$

We remark that $F = p^*(f)$, where $p : S^2 \to \mathbb{R}P^2$ is the standard double covering map. We compute the Jacobian dF:

$$dF_{(x,y,z)} = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

Strictly speaking, the jacobian dF_p at $p = (x, y, z) \in S^2$ is the restriction of the above matrix to the 2-dimensional subspace $T_p(S^2)$ consisting of vectors orthogonal to p. We note that the determinant of the above 3 by 3 matrix is $2xyz$, so that F is definitely an immersion when x, y and z are all non-zero.

It remains to study the cases where one of them vanishes. Let us say that $z = 0$*. Then*

$$dF_{(x,y,0)} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ y & x & 0 \end{pmatrix}.$$

Moreover, $T_{(x,y,0)}$ *is the span of* $-ye_1 + xe_2$ *and* e_3*, and their images under* dF *are given by:*

$$\begin{aligned} dF_{(x,y,0)}(-ye_1 + xe_2) &= (x^2 - y^2)e_3 \\ dF_{(x,y,0)}(e_3) &= ye_1 + xe_2. \end{aligned}$$

Thus $dF_{(x,y,0)}$ *only fails to be an immersion when* $y = \pm x$*. And we could repeat the discussion in a similar way to the remaining two cases* $x = 0$ *and* $y = 0$*, obtaining that* F *is an immersion except at the following points:*

$$\begin{aligned} (x, y, z) =& (0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}) \text{ or} \\ & (\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}) \text{ or} \\ & (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0). \end{aligned}$$

These 12 *points come in* 6 *pairs of antipodal points, so that* f *fails to be an immersion only at the corresponding* 6 *points.*

Problem 4: (Immersions and submersions)

a) Show that an immersion between two n-dimensional manifolds is an open map.

Let $f : M \to N$ *be an immersion where* M *and* N *have the same dimension* n*. We want to show that the image* $f(U)$ *of a (non-empty) open set* U *is open in* N*. Let* $q \in f(U)$*, and let* $p \in M$ *such that* $f(p) = q$*. The constant rank theorem applies, and gives a neighborhood* U' *of* p *and a neighborhood* V' *of* q *such that* f *restricted to* U' *is a diffeomorphism from* U' *onto* V'*. But by possibly intersecting* U' *with* U*, one can without loss of generality assume that* $U' \subseteq U$*. Thus* V' *is a neighborhood of* q *contained in* $f(U)$*. Since* $q \in f(V)$ *was arbitrary, this shows that* f *is an open map.*

b) Show that if $f : M \to N$ is a surjective submersion, then the pull-back map $f^* : C^\infty(N, T^*(N)) \to C^\infty(M, T^*(M))$, from the space of 1-forms on N to the space of 1-forms on M is injective.

Let β be a 1-form on N whose pullback $f^*(\beta)$ under a submersion $f : M \to N$ vanishes. Let $q \in N$, and let $p \in M$ such that $f(p) = q$. Let $Y \in T_q(N)$. Since f is a submersion, it follows that we can find a vector $X \in T_p(M)$ such that $f_*(X) = Y$. We then have

$$f^*(\beta)_p(X) = \beta_q(f_*(X)).$$

But the left-hand side vanishes, by hypothesis, so that $\beta_q(Y) = 0$. Y was arbitrary, so we have proved that $\beta = 0$. Hence the pullback map f^* on smooth 1-forms is injective, if f is a surjective submersion.