Solution to Homework 3:

1) Hyperbolic Space

a) Since $\tau : I^+(s) \to I^+(s)$ is an involution (i.e. $\tau^2 = Id$), it suffices to show that $\tau(H^n) \subseteq B^n$ and $\tau(B^n) \subseteq H^n$.

- if $x \in H^n$, then $\tau(x) = s + \frac{p-s}{\|p-s\|^2} = \frac{1}{2(x^0+1)}\left(-(x^0+1), x^1, \dots, x^n\right)$

$\Rightarrow x^0(\tau(x)) = -\frac{1}{2}$ and $\langle \tau(x), \tau(x)\rangle = -\frac{1}{2(x^0+1)} < 0$,

so $\tau(x) \in B^n$

- if $y \in B^n$, then $\tau(y) = s + \frac{y-s}{\|y-s\|^2} = \frac{1}{1-4\sum_{i=1}^n (y^i)^2}\left(1+4\sum_{i=1}^n (y^i)^2, 4y^1, \dots, 4y^n\right)$

$\Rightarrow \langle \tau(y), \tau(y)\rangle = -1$

b) Let $f = \tau|_{B^n} : B^n \to H^n$. We want to compute $f^*(g)$.

$$f^*(x^0) = \frac{1+4\sum_{i=1}^n (y^i)^2}{1-4\sum_{j=1}^n (y^j)^2}$$

$$\Rightarrow f^*(dx^0) = \frac{16\sum_{i=1}^n y^i dy^i}{\left(1-4\sum_{j=1}^n (y^j)^2\right)^2}$$

$$f^*(x^i) = \frac{4y^i}{1-4\sum_{j=1}^n (y^j)^2}, \quad 1 \le i \le n$$

$$\Rightarrow f^*(dx^i) = \frac{4}{\left(1-4\sum_{j=1}^n (y^j)^2\right)^2}\left[\left(1-4\sum_{k=1}^n (y^k)^2\right)dy^i + 8y^i\sum_{k=1}^n y^k dy^k\right]$$

Therefore, after simplifying, we obtain:

$$f^*(g) = \frac{16 \sum_{i=1}^{n} (dy^i)^2}{\left(1 - 4\sum_{j=1}^{n} (y^j)^2\right)^2} \quad \text{on } B^n$$

But B^n has radius $\frac{1}{2}$, so we introduce $y^i = \frac{1}{2} u^i$ (then $\underline{u} = (u^1, \dots, u^n)$ is in the open ball of radius 1).

Then

$$\boxed{\text{(The pullback of) } f^*(g) = \frac{4 \sum_{i=1}^{n} (du^i)^2}{\left(1 - \sum_{j=1}^{n} (u^j)^2\right)^2} \quad \text{on the open unit ball.}}$$

This is the hyperbolic metric on the (open) unit ball in $\mathbb{R}^n$.

2) (De Sitter and Anti de Sitter Spaces)

a) dS^n:

Let $O(n,1) = \{ G \in GL_{n+1}(\mathbb{R}) \mid \langle Gv, Gw \rangle_1 = \langle v, w \rangle_1 ,$ for any $v, w \in \mathbb{R}^{n+1} \}$

$O(n,1)$ is a noncompact (why?) Lie group of dimension $\frac{n(n+1)}{2}$.

Fact 1: $O(n,1)$ acts isometrically on dS^n.

Proof: $O(n,1)$ acts isometrically on $(\mathbb{R}^{n+1}, \langle -,- \rangle_1)$, since

$$\langle (L_G)_* X_p , (L_G)_* Y_p \rangle_{1;G(p)} = \langle G X_p , G Y_p \rangle_1 = \langle X_p , Y_p \rangle_{1;p} ,$$

where $L_G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denotes the natural action of G on $\mathbb{R}^{n+1}$.

Moreover, if $x \in dS^n$, then $\langle x, x \rangle_1 = 1$, and $\langle L_G(x), L_G(x) \rangle_1 = \langle x, x \rangle_1 = 1$, so that $L_G(dS^n) \subseteq dS^n$, for any $G \in O(n,1)$.

<u>Fact 2:</u> $O(n,1)$ acts transitively on dS^n.

<u>Proof:</u> Let $x = (x^0, x^1, \ldots, x^n) \in dS^n$

Then $-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = 1$.

$\Rightarrow (x^1, \ldots, x^n) \in S^{n-1}\left(\sqrt{1+(x^0)^2}\right)$

But we know that $O(n) \leq O(n,1)$ acts transitively on $S^{n-1}(R)$ for a fixed radius $R>0$. Hence $\exists\, G_1 \in O(n) \leq O(n,1)$ such that

$G_1(x) = \left(x^0, \sqrt{1+(x^0)^2}, 0, \ldots, 0\right)$

Let
$$G_2 = \begin{pmatrix} \begin{pmatrix} \sqrt{1+(x^0)^2} & x^0 \\ x^0 & \sqrt{1+(x^0)^2} \end{pmatrix} & O_{2,n-1} \\ O_{n-1,2} & O_{n-1,n-1} \end{pmatrix}$$

Then $G_2 \in O(n,1)$ and $G_2(e_1) = G_1(x)$

$\Rightarrow G_2^{-1} G_1(x) = e_1$

Hence, for any $x \in dS^n$, there is a $G \in O(n,1)$ such that $G(x) = e_1$ (Remark: G is not unique).

This shows that $O(n,1)$ acts transitively on dS^n.

(de Sitter and Antide Sitter)

Using facts 1 and 2, it suffices to show that g is Lorentzian at $e_1 \in dS^n$. $T_{e_1}(dS^n) = \text{span}(e_0, e_2, e_3, \dots, e_n)$, and with respect to the basis $e_0, e_2, e_3, \dots, e_n$, we see that:

$$g_{e_1} = \left.\begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}\right\} n$$

Hence g, the induced metric on dS^n, is Lorentzian.

- AdS^n:

Let $O(n-1,2) = \{ G \in GL_{n+1}(\mathbb{R}) \mid \langle Gv, Gw \rangle_2 = \langle v, w \rangle_2$ for any $v, w \in \mathbb{R}^{n+1} \}$

Similarly, we have:

Fact 1': $O(n-1,2)$ acts isometrically on AdS^n.

Fact 2': $O(n-1,2)$ acts transitively on AdS^n.

Sketch of proof of Fact 2'. There is a $G_1 \in O(2) \leq O(n-1,2)$, such that G_1 maps a given $x \in AdS^n$ to:

$$G_1(x) = \left(\sqrt{(x^0)^2 + (x^1)^2}, 0, x^2, \dots, x^n \right)$$

There is a $G_2 \in O(n-1) \leq O(n-1,2)$ such that:

$$G_2 G_1(x) = \left(\sqrt{(x^0)^2 + (x^1)^2}, 0, \sqrt{(x^2)^2 + \dots + (x^n)^2}, 0, \dots, 0 \right)$$

Let $G_3 = \begin{pmatrix} \sqrt{(x^0)^2+(x^1)^2} & 0 & \sqrt{(x^2)^2+\cdots+(x^n)^2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \sqrt{(x^2)^2+\cdots+(x^n)^2} & 0 & \sqrt{(x^0)^2+(x^1)^2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \\ \vdots & & & \ddots & \vdots \end{pmatrix}$

$\Rightarrow \quad G_3 e_0 = G_2 G_1(x) \qquad\qquad (G_3 \in O(n-1,2))$

$\Rightarrow \quad G_3^{-1} G_2 G_1(x) = e_0$, finishing the proof of fact 2'.

Now it suffices to show that g_{e_0} is Lorentzian, which is easy.

b) For dS^n, using an appropriate $G \in O(n,1)$, one can always arrange for the spacelike hyperplane V to be of the form $\{x^0 = c\}$. More precisely, there is a $G \in O(n,1)$ such that $G(V) = \{x^0 = c\}$.

$$\Rightarrow \quad G(V \cap dS^n) = \{x^0 = c\} \cap dS^n = \{(c, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} \mid (x^1)^2 + \cdots + (x^n)^2 = 1 + c^2\}$$

$$\overset{\text{isometric}}{\simeq} S^{n-1}\left(\sqrt{1+c^2}\right)$$

But G is an isometry, so $V \cap dS^n \overset{\text{isometric}}{\simeq} S^{n-1}(R)$, for some $R \geq 1$ which depends on V.

For AdS^n, one can similarly arrange that the hyperplane V is of the form $\{x^0 = c\}$ using a $G \in O(n-1, 2)$.

$$\Rightarrow G(V \cap AdS^n) = \{x^0 = c\} \cap AdS^n$$

$$= \{(c, x^1, x^2, \ldots, x^n) \in \mathbb{R}^{n+1} \mid -(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 = -1 + c^2\}$$

Then * if $c^2 > 1$, we get dS^{n-1} up to rescaling

* if $c^2 < 1$, we get two copies of H^{n-1}, up to rescaling.

* if $c^2 = 1$, the intersection becomes degenerate:

$$\left(\text{~~of~~ the form } -(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 = 0 \right).$$

All 3 cases do occur.

c) Define $f : dS^n \longrightarrow \mathbb{R} \times S^{n-1}$ by:

$$f(x^0, x^1, \ldots, x^n) = \left(x^0, \frac{1}{\sqrt{\sum_{i=1}^{n} (x^i)^2}} (x^1, \ldots, x^n) \right)$$

Define $h : AdS^n \longrightarrow S^1 \times \mathbb{R}^{n-1}$ by:

$$h(x^0, x^1, x^2, \ldots, x^n) = \left(\frac{1}{\sqrt{(x^0)^2 + (x^1)^2}} (x^0, x^1), (x^2, \ldots, x^n) \right).$$

You can check that f & h are diffeomorphisms, by explicitly computing their inverses, say.

Alternative (and easier) solution of 2) a):

Let $x \in dS^n \Rightarrow -(x^0)^2 + (x^1)^2 + \dots + (x^n)^2 = 1$

Let $\rho = \sqrt{(x^1)^2 + \dots + (x^n)^2}$

We want to show that $T_x(dS^n)$ is Lorentzian.

$T_x(dS^n) = \{ y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle_1 = 0 \}$

Let $v = \begin{pmatrix} \rho \\ \frac{x^0 x^1}{\rho} \\ \frac{x^0 x^2}{\rho} \\ \vdots \\ \frac{x^0 x^n}{\rho} \end{pmatrix}$. Then $v \in T_x(dS^n)$ and $\langle v, v \rangle_1 = -1$ (i.e. v is timelike).

We then consider $\operatorname{span}(x, v)^{\perp} = \{ y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle_1 = 0$ and $\langle v, y \rangle_1 = 0 \}$

$\langle y, v \rangle_1 = 0 \Rightarrow y^0 x^0 = x^1 y^1 + \dots + x^n y^n \qquad (1)$

$\langle y, x \rangle_1 = 0 \Rightarrow y^0 \rho^2 = x^0 (x^1 y^1 + \dots + x^n y^n) \qquad (2)$

(1) & (2) imply that $y^0 \rho^2 = (x^0)^2 y^0 \Rightarrow y^0(-(x^0)^2 + \rho^2) = 0$
$\Rightarrow \boxed{y^0 = 0}$

So (1) & (2) are equivalent to: $\begin{cases} y^0 = 0, \text{ and} \\ x^1 y^1 + \dots + x^n y^n = 0 \end{cases}$

This shows that $\langle -, - \rangle_1$ restricted to $T_x(dS^n)$ has signature $(n-1, 1)$, i.e. is Lorentzian.

(Similarly AdS^n can be shown to be Lorentzian).

Remark on 2) b):

One can solve 2) b) in an easier (and more elementary way) by writing down the equation for a general spacelike hyperplane in $\mathbb{R}^{n+1}$ and then intersecting that hyperplane with dS^n (or AdS^n).

For dS^n: A general spacelike hyperplane H can be written as the set of $x \in \mathbb{R}^{n+1}$ such that:

$$-v^0 x^0 + v^1 x^1 + \cdots + v^n x^n = c, \qquad (*)$$

where $v = \begin{pmatrix} v^0 \\ v^1 \\ \vdots \\ v^n \end{pmatrix}$ is timelike and $\langle v, v \rangle_1 = -1$ and c is some real number.

Hence if we call the hyperplane defined by $(*)$ H, then

$$H \cap dS^n = \{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_1 = 1 \text{ and } \langle x, v \rangle_1 = c \}$$

One can then show that $H \cap dS^n$ is a sphere in H.

~~centered at ([illegible] , [illegible] , 0) of radius~~

~~$R = \sqrt{\left(1 - \frac{c}{v^0}\right)^2 - \left(\frac{c}{v^0}\right)^2}$~~