Solution to Homework 4:

1) a) $D \to$ linear connection

$$T^D(X,Y) = D_X Y - D_Y X - [X,Y]$$

$$T^D(fX,Y) = f D_X Y - f D_Y X - L_Y(f)X + L_Y(f)X - f[X,Y]$$
$$= f\, T^D(X,Y) \qquad \forall f \in C^\infty(M)$$

$\Rightarrow T^D$ is linear (over $C^\infty(M)$) in the first argument.

But $T^D(X,Y) = -T^D(Y,X)$, so T^D is a $(2,1)$ tensor.

b) Let $A = D' - D$ (i.e. $A(X,Y) = D'_X Y - D_X Y$)

Then $A(fX,Y) = f A(X,Y) \quad \forall f \in C^\infty(M)$

$$A(X, fY) = f A(X,Y) + L_X(f) Y - L_X(f) Y = f A(X,Y)$$

So A is a $(2,1)$-tensor.

- Suppose D & D' have the same parametrized geodesics. Then $A(X,X) = D'_X X - D_X X = 0 \quad \forall X \in \Gamma(M, TM)$

$$\Rightarrow A(X,Y) + A(Y,X) = \tfrac{1}{2}\big(A(X+Y, X+Y) - A(X-Y, X-Y)\big)$$
$$= 0$$

So A is skew-symmetric

- Conversely, if A is skew-symmetric, and $\gamma(t)$ is a geodesic for D, then

$$D'_{\dot\gamma(t)} \dot\gamma(t) = D_{\dot\gamma(t)} \dot\gamma(t) + A(\dot\gamma(t), \dot\gamma(t))$$
$$= 0$$

So $\gamma(t)$ is a geodesic for D'.

c) $\tilde{g} = e^{f} g$

Apply the Koszul formula for $\tilde{\nabla}$, the L-C connection of $\tilde{g}$.

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) = L_X(\tilde{g}(Y,Z)) + L_Y(\tilde{g}(X,Z)) - L_Z(\tilde{g}(X,Y)) + \tilde{g}([X,Y],Z) - \tilde{g}([X,Z],Y) - \tilde{g}([Y,Z],X)$$

$$\Rightarrow \boxed{g(\tilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2} X(f)\, g(Y,Z) + \frac{1}{2} Y(f)\, g(X,Z) - \frac{1}{2} Z(f)\, g(X,Y)} \quad (*)$$

Let $\gamma(t)$ be a null geodesic for ∇. We claim that you can reparametrize γ, so that $\tilde{\gamma}(\tau) = \gamma(H(\tau))$, is a null geodesic for $\tilde{\nabla}$.

Apply $(*)$ with $X = Y = \frac{d\tilde{\gamma}}{d\tau}$, and Z arbitrary vector field along $\tilde{\gamma}(\tau)$. Let $\tilde{f}(\tau) = f(H(\tau))$.

$$\Rightarrow 2\tilde{g}\left(\tilde{\nabla}_{\frac{d\tilde{\gamma}}{d\tau}} \frac{d\tilde{\gamma}}{d\tau}, Z\right) = g\left(\frac{dH}{d\tau} \nabla_{\gamma'(H(\tau))}\left(\frac{dH}{d\tau}\gamma'(H(\tau))\right), Z\right) + \frac{d\tilde{f}}{d\tau}\, g\left(\frac{dH}{d\tau}\gamma'(H(\tau)), Z\right).$$

We want the left-hand side to vanish, for any Z, so we impose:

$$\frac{d^2 H}{d\tau^2} + \frac{d\tilde{f}}{d\tau}\frac{dH}{d\tau} = 0$$

Integrating, we get:

$$\frac{dH}{d\tau} = C\, e^{-f(H(\tau))}$$

Set $C = 1$, and look for solutions of:

$$\frac{dH}{d\tau} = e^{-f(H(\tau))}$$

This is non-linear, but let $G = H^{-1}$, so

$$\frac{1}{\frac{dG}{dt}} = e^{-f(t)}$$

$$\Rightarrow \quad \frac{dG}{dt} = e^{f(t)}$$

$$\Rightarrow \quad G(t) = \int e^{f(t)}\, dt$$

We thus get that if $G(t) = \int e^{f(t)}\, dt$, and $H = G^{-1}$, then the curve $\gamma(H(\tau))$ is a null geodesic for $\hat{\nabla}$ if $\gamma(t)$ is a null geodesic for ∇.

d) If ∇ is the L-C connection of g, then

$$(\nabla_X g)(Y,Z) = L_X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,$$

since ∇ is metric.

e) If ∇ is the L-C connection of g, then

$$(\nabla df)(X,Y) = X(Y(f)) - (\nabla_X Y)(f)$$

So $(\nabla df)(X,Y) - (\nabla df)(Y,X) = [X,Y](f) - (\nabla_X Y - \nabla_Y X)(f)$

$$= -T^{\nabla}(X,Y)$$

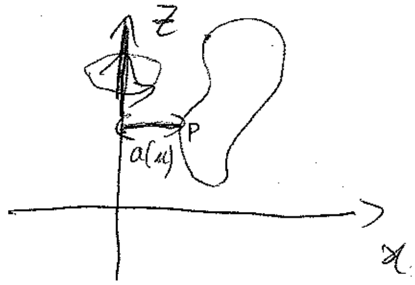
$$= 0$$

In local coordinates x^i, $1 \leq i \leq n$:

$$\nabla df = \left(\partial_{x^i}\partial_{x^j} f - \Gamma_{ij}{}^{k}\,\partial_{x^k} f\right) dx^i \otimes dx^j$$

(using the Einstein summation convention).

2) a) $g = du^2 + a^2(u)\, d\theta^2$ (I),
where $a(u)$ is the distance of the point (u,θ) to the axis of rotation.



b) Using (I) and the Koszul formula, we compute

$\nabla_{\partial_u} \partial_u \equiv 0 \Rightarrow$ meridians $\theta =$ constant are geodesics

$\nabla_{\partial_u} \partial_\theta = \nabla_{\partial_\theta} \partial_u = \frac{a'}{a} \partial_\theta$

$\nabla_{\partial_\theta} \partial_\theta = -a a' \partial_u \Rightarrow$ parallels $u =$ constant are geodesics if $a'(u) = 0$.

We are interested in $t \xrightarrow{\gamma} (u(t), \theta(t))$ satisfying the geodesic equation (and parametrized by arc length):

$\nabla_{\gamma'(t)} \gamma'(t) \equiv 0$

i.e.: $0 = u'' \partial_u + \theta'' \partial_\theta + (u')^2 \nabla_{\partial_u} \partial_u + u'\theta' (\nabla_{\partial_u} \partial_\theta + \nabla_{\partial_\theta} \partial_u)$
$+ (\theta')^2 \nabla_{\partial_\theta} \partial_\theta$

$= (u'' - (\theta')^2 a a') \partial_u + (\theta'' + \frac{2u'\theta' a'}{a}) \partial_\theta$

$$\Leftrightarrow \begin{cases} u'' - (\theta')^2 a a' = 0 & (1) \\ \theta'' + \frac{2u'\theta' a'}{a} = 0 & (2) \end{cases}$$

Multiply (2) by a^2 and integrate:

$$(a^2\theta')' = 0$$

$\Rightarrow a^2\theta' =$ constant, i.e. $\boxed{a^2(u(t))\,\theta'(t) = C.}$

Moreover, since $g(\gamma'(t), \gamma'(t)) \equiv 1$, we get:

$$\boxed{\left(\frac{du}{dt}\right)^2 + a^2(u(t))\left(\frac{d\theta}{dt}\right)^2 \equiv 1}$$

c) The equation $a^2(u(t))\,\theta'(t) = C$ can be interpreted as $a(u)\cos(\alpha) = C$, where α is the angle between $\gamma'(t)$ and ∂_θ.

Assuming that the geodesic is parametrized by length, we get a new system:

$$\left(\frac{du}{dt}\right)^2 + a^2(u(t))\left(\frac{d\theta}{dt}\right)^2 = 1,$$

$$\text{and} \quad a^2(u(t))\frac{d\theta}{dt} = C.$$

One proves easily that this system is equivalent to the first one. Eventually reversing the parametrization, we can assume that $\frac{d\theta}{dt} > 0$, so that C is positive. Since

$$a^2(u(t)).\left(\frac{d\theta}{dt}\right)^2 \leq 1,$$

we always have $a\big(u(t)\big) \geq C$, with equality at time s if and only if $\frac{du}{dt}(s) = 0$. Generically, the geodesic oscillates between two consecutive parallels satisfying to $a(u) = C$ (fig.i), to which it is tangent. If one of these parallels is extremal (that is $a'(u) = 0$), the geodesic is asymptotic to this parallel (fig.ii), which is itself a geodesic.

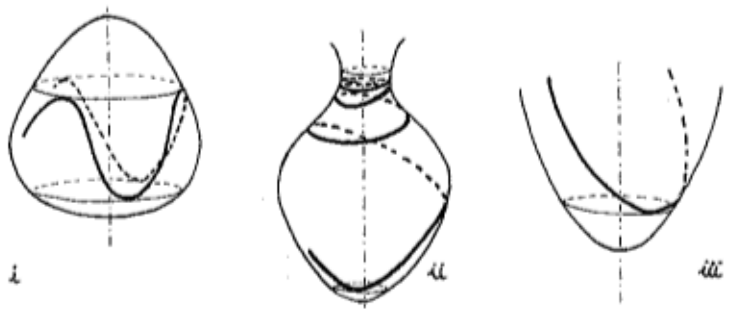


Fig. B.7. Geodesics on a revolution surface