Solution to Homework 5:

1) $H = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

$$ds^2 = g = \frac{1}{y^2}(dx^2 + dy^2)$$

a) Use the formula:

$$\Gamma_{ij}^{\ \ k} = \frac{1}{2} g^{kl} \left(\partial_{x^j} g_{il} + \partial_{x^i} g_{jl} - \partial_{x^l} g_{ij} \right)$$

$$\Rightarrow \Gamma_{11}^{\ \ 2} = \frac{1}{y}$$

$$\Gamma_{22}^{\ \ 2} = -\frac{1}{y}$$

$$\Gamma_{12}^{\ \ 1} = \Gamma_{21}^{\ \ 1} = -\frac{1}{y}$$

All other $\Gamma_{ij}^{\ \ k}$ being 0.

b) The geodesic equation is:

$$\frac{d^2 x^k}{dt^2} + \Gamma_{ij}^{\ \ k} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$$

which gives:
$$\ddot{x} y = 2 \dot{x} \dot{y}$$
$$\ddot{y} y = (\dot{y})^2 - (\dot{x})^2$$

Consider the curve $\boxed{\mathrm{I}}$ $\begin{cases} x = c \\ y = e^t \end{cases}$, $t \in \mathbb{R}$ $\quad$ (c = constant)

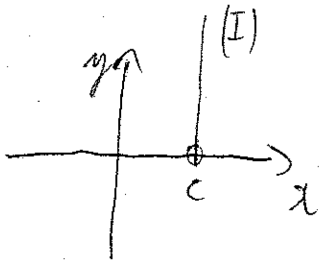
One can check that it is a geodesic parametrized by arc length.

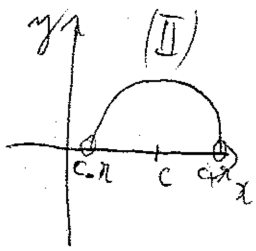
Next consider: $\boxed{\mathrm{II}}$ $\begin{cases} x = -r \tanh(t) + c \\ y = r \,\mathrm{sech}(t) \end{cases}$, $t \in \mathbb{R}$ $\quad$ ($c \in \mathbb{R}$, $r > 0$)

One can also check that it is a geodesic parametrized by arc length.

The first type of geodesics (I) are vertical rays (with $x=c$).

The second type (II) are upper semicircles centered at $(c,0)$ of radius $r>0$.



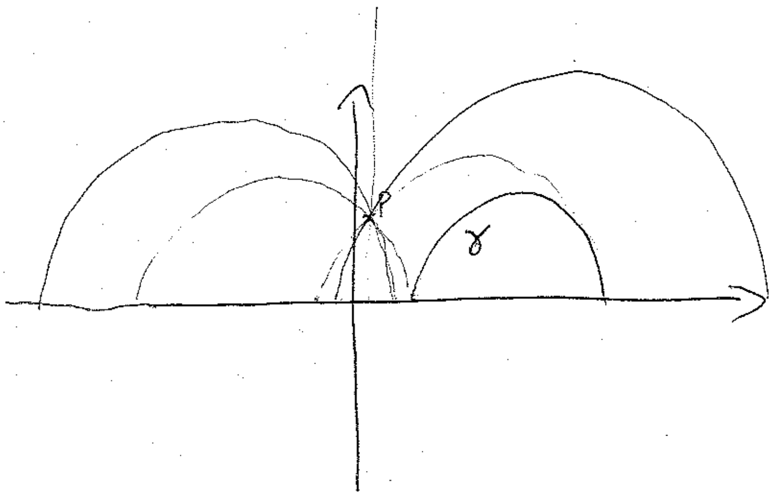


These are, up to affine reparametrization, all the geodesics, because at any point $p \in H$, and for any $v \in T_p(H)$, $\|v\|_p = 1$, there is a geodesic either of type (I) or (II) passing through p with velocity v.

c) $\int_{t_0}^{\infty} dt = +\infty$ & $\int_{-\infty}^{t_0} dt = +\infty$,

so the geodesics have infinite length in either direction.

d)



Hard to draw, but such geodesics are parametrized by an open interval (a,b), $a<b$, and the limiting geodesics corresponding to a and b are asymptotic to γ at the left end resp. right end of γ at infinity.

2) a) $$\frac{L(\exp\circ c)}{L(c)} = \frac{\int_0^1 \| d\exp_m(v + t(w-v))(w-v)\| \, dt}{\|w-v\|_m}$$

$$= \int_0^1 \left\| d\exp_m(v + t(w-v))\left(\frac{w-v}{\|w-v\|_m}\right)\right\| dt$$

$$\longrightarrow \int_0^1 1 \, dt = 1 \quad , \text{ as } v, w \to 0,$$

because $\lim_{v,w\to 0} d\exp_m(v + t(w-v)) = d\exp_m(0) = Id$

b) $$\frac{L(\exp_m^{-1}\circ \gamma_{v,w})}{L(\gamma_{v,w})} = \frac{\int_0^D \| d\exp_m^{-1}(\gamma_{v,w}(t))(\gamma_{v,w}'(t))\|_m \, dt}{\int_0^D \|\gamma_{v,w}'(t)\| \, dt}$$

$$= \frac{1}{D}\int_0^D \| d\exp_m^{-1}(\gamma_{v,w}(t))(\gamma_{v,w}'(t))\|_m \, dt$$

$$\longrightarrow \| d\exp_m^{-1}(m)(\gamma_{v,w}'(0))\|_m \quad , \text{ as } v, w \to 0$$

(and therefore $D \to 0$)

$$= \|\gamma_{v,w}'(0)\|_m$$

$$= 1 \quad ,$$ since $d\exp_m^{-1}(m) = Id$ (and the geodesic has unit length)

c) $$\frac{\|w-v\|_m}{D} = \left\| \frac{1}{D}\int_0^D c_{v,w}'(t)\, dt \right\|_m$$

$$\longrightarrow \lim_{v,w\to 0} \| c_{v,w}'(0)\|_m$$

$$= 1 \quad , \text{ as } v, w \to 0,$$

where $D = d(\exp_m(v), \exp_m(w))$

$= L(\gamma_{v,w})$

3) a) Assume $\Theta_t \circ \Psi_s = \Psi_s \circ \Theta_t \quad \forall s, t$

$$(\Theta_t \circ \Psi_s)^*(f) = \Psi_s^* \circ \Theta_t^*(f)$$

$$\Rightarrow \frac{\partial^2}{\partial s \partial t} (\Theta_t \circ \Psi_s)^*(f)\Big|_{(s,t)=(0,0)} = YX(f)$$

Similarly, $\frac{\partial^2}{\partial s \partial t} (\Psi_s \circ \Theta_t)^*(f)\Big|_{(s,t)=(0,0)} = XY(f)$

$\Rightarrow [X,Y] = 0$ (since $f \in C^\infty(M)$ was arbitrary).

Conversely, assume $[X,Y] = 0$.

Note that for a fixed t, $s \mapsto \Theta_t \circ \Psi_s \circ \Theta_{-t}$ is a flow.

Consider $\frac{d}{ds} (\Theta_t \circ \Psi_s \circ \Theta_{-t})^*(f)\Big|_{s=0} = \Theta_{-t}^* \circ Y \circ \Theta_t^*$

Thus $s \mapsto \Theta_t \circ \Psi_s \circ \Theta_{-t}$ is the flow for

$\Theta_{-t}^* \circ Y \circ \Theta_t^*$.

Moreover $\frac{d}{dt} (\Theta_{-t}^* \circ Y \circ \Theta_t^*) = - [X,Y] = 0$, by assumption.

So $\Theta_{-t}^* \circ Y \circ \Theta_t^* = \Theta_0^* \circ Y \circ \Theta_0^* = Y$

Hence by uniqueness of flows,

Flow of $(\Theta_{-t}^* \circ Y \circ \Theta_t^*)$ = Flow of (Y)

i.e. $\Theta_t \circ \Psi_s \circ \Theta_{-t} = \Psi_s$.

$\Rightarrow \Theta_t \circ \Psi_s = \Psi_s \circ \Theta_t$.

QED.

3) b) Assume that $v, w \in \mathfrak{g}$, and $[v, w] = 0$.

v & w can be thought of as left-invariant vector fields on G.

Let A_s, B_t be the corresponding flows of v, w resp.

$A_t(e)\, B_t(e) = (B_t \circ A_t)(e)$ since B_t is a left-invariant flow.

But since $[v, w] = 0$, then $B_s \circ A_t = A_t \circ B_s$ $\forall s, t$ by 3)a),

and so $t \mapsto B_t \circ A_t$ is a flow, in fact it is the flow of $v + w$ (because $t_1 + t_2 \mapsto B_{t_1} \circ A_{t_1} \circ B_{t_2} \circ A_{t_2}$, so it is enough to differentiate at $t = 0$)

But $A_1(e)\, B_1(e) = (B_1 \circ A_1)(e)$,

so $\exp(v) \exp(w) = \exp(v + w)$.

4) a) Let $0 \neq \tilde{\Omega} \in \Lambda^n T_e^*(G) \simeq \Lambda^n \mathfrak{g}^* \simeq \mathbb{R}$ (non-vanishing)

Then extent $\tilde{\Omega}$ to a left-invariant n-form by:

$$\Omega_g = (L_{g^{-1}})^* \tilde{\Omega}$$

Ω is then a smooth n-form of G. Moreover:

$$\Omega_{g_1 g} = (L_{(g_1 g)^{-1}})^* \tilde{\Omega} = (L_{g_1^{-1}})^* L_{g^{-1}}^* \tilde{\Omega} = (L_{g_1^{-1}})^* \Omega_g$$

$$\Rightarrow (L_{g_1})^* \Omega = \Omega \quad \forall g_1 \in G.$$

4) b) $GL(n,\mathbb{R})$ has standard coordinates

x_{ij}, $\det(x_{ij}) \neq 0$.

Define $\Omega = \dfrac{1}{\det(x_{ij})^n} dx^{11} \wedge dx^{21} \wedge \dots \wedge dx^{n1} \wedge dx^{12} \wedge dx^{22} \wedge \dots \wedge dx^{n2} \wedge \dots$
$\dots \wedge dx^{1n} \wedge dx^{2n} \wedge \dots \wedge dx^{nn}$

Let $g \in GL(n,\mathbb{R})$.

$$(L_g)^*(dx^{11} \wedge \dots \wedge dx^{nn}) = \det(g)^n$$

$$(L_g)^* \frac{1}{\det(x_{ij})^n} = \frac{1}{\det(g)^n} \frac{1}{\det(x_{ij})^n}$$

So $(L_g)^*(\Omega) = \Omega \quad \forall g \in GL(n,\mathbb{R})$.

c) Let h_0 be any smooth metric on M.

Define $h = \int_G g^*(h_0)\, \Omega_G$ (which is also a smooth metric since action is smooth and Ω_G is smooth).

where Ω_G is the Haar n-form of G ($\dim G = n$), normalized so that $\int_G \Omega_G = 1$.

Let $g_1 \in G$. Then

$$g_1^*(h) = \int_G g_1^* g^*(h_0)\ g^*\Omega_G$$

$$= \int_G (g g_1)^*(h_0)\ \Omega_G$$

$$= h,$$

so h is left-invariant.

Optional problem

a) elements of $\mathfrak{g}$ are identified with linear functions on $\mathfrak{g}^*$, i.e. with coordinates. The Poisson bracket in these coordinates is

$$\{f,g\} = \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} [x_i, x_j] \qquad (*)$$

Write $X_f = \sum_{i,j} \frac{\partial f}{\partial x_i} [x_i, x_j] \frac{\partial}{\partial x_j}$ – vector field on $\mathfrak{g}^*$.

Direct calculation: $[X_f, X_g] = X_{\{f,g\}}$ where the bracket is the vector field bracket. Jacobi identity follows.

Another proof of the Jacobi identity: The formula $(*)$ proves it for linear functions. Using part (b) we can show that it holds for polynomial ~~quasi~~ functions. Since polynomials are dense in $C^\infty(\mathfrak{g}^*)$, the Jacobi identity follows ~~from~~ for $C^\infty(\mathfrak{g}^*)$.

b) Trivial consequence of the Leibnitz rule for functions.

c) Let $A \in gl_n(\mathbb{R})$. Under the identifications

$$d\chi_k\big|_A = kA^{k-1} \in gl_n(\mathbb{R})$$

Hence $\{\chi_k, q\} = tr\left(A\left[kA^{k-1}, dq|_A\right]\right) =$

$= tr\left(\left[A, kA^{k-1}\right] \cdot dq|_A\right) = 0$