Solution to Homework 6:

1) Fubini - Study metric

Consider $\begin{array}{c} \mathbb{C}^{n+1} - \{0\} \\ \downarrow p \\ S^{2n+1} \\ \downarrow q \\ \mathbb{CP}^n \end{array}$, where $p(z) = \frac{z}{\|z\|}$, and q is the Hopf map, with fibres circles.

We denote by $\langle z, w \rangle$ the hermitian inner product on $\mathbb{C}^{n+1}$: $\langle z, w \rangle = \sum_{\alpha=0}^{n} z^\alpha \overline{w^\alpha}$

and by (z, w) the corresponding inner product:

$$(z, w) = \mathrm{Re}(\langle z, w \rangle) = \frac{1}{2}\left(\langle z, w \rangle + \langle w, z \rangle\right)$$

- We also denote by $\langle z, dz \rangle = \sum_{\alpha=0}^{n} z^\alpha d\overline{z^\alpha}$

$$(z, dz) = \mathrm{Re}\langle z, dz \rangle$$

$$\langle dz, dz \rangle = (dz, dz) = \sum_{\alpha=0}^{n} dz^\alpha \odot d\overline{z^\alpha},$$

where $\odot$ is the symmetrized tensor product.

Strategy: compute $p^*(q^* g_{F-S})$ in 2 steps (g_{F-S} = Fubini-Study metric)

Step 1: compute $g_1 = p^*(g_{S^{2n+1}})$

Step 2: compute $p^*(q^* g_{F-S}) = g_1 - \frac{g_1(X) \otimes g_1(X)}{g_1(X, X)}$

where X is a generating vector field of the $U(1)$ action on S^{2n+1}.

Step 1: $g_1 = \left(d\left(\frac{z}{\|z\|}\right), d\left(\frac{z}{\|z\|}\right)\right)$

$$= \frac{\|z\|^2 \cdot \|dz\|^2 - 2(z, dz)^2 + (z, dz)^2}{\|z\|^4}$$

$$= \frac{\|z\|^2 \cdot \|dz\|^2 - (z, dz)^2}{\|z\|^4}$$

Step 2: $\lambda \in U(1)$ acts on $\mathbb{C}^{n+1} - \{0\}$ by:

$\lambda \cdot z = \lambda z$

Writing $\lambda_t = e^{it}$, and differentiating φ_t at $t = 0$, where $\varphi_t(z, \bar{z}) = (\lambda_t z, \bar{\lambda}_t \bar{z})$, we get a real vector field X on $\mathbb{C}^{n+1} - \{0\}$ given by:

$X = i\left(z^\alpha \partial_{z^\alpha} - \bar{z}^\alpha \partial_{\bar{z}^\alpha}\right)$. ~~But we actually take $X = \frac{\hat{X}}{\|z\|}$, to get $dp(X)$ well defined on S^{2n+1}~~

Then $p^*(q^* g_{F\text{-}S}) = g_1 - \frac{g_1(X) \otimes g_1(X)}{g_1(X, X)}$

But $g_1(X, X) = 1$ and $g_1(X) = \frac{(iz, dz)}{\|z\|^2}$

$$\Rightarrow \quad p^*(q^* g_{F\text{-}S}) = \frac{\|dz\|^2}{\|z\|^2} - \left(\frac{(z, dz)^2 + (iz, dz)^2}{\|z\|^4}\right)$$

But $(z, dz)^2 + (iz, dz)^2 = \frac{1}{4}\left(\langle z, dz\rangle + \langle dz, z\rangle\right)^2 + \frac{1}{4}\left(i\langle z, dz\rangle - i\langle dz, z\rangle\right)^2$

$$= \langle dz, z\rangle \otimes \langle z, dz\rangle$$

So $p^*(q^* g_{F-S}) = \dfrac{\|dz\|^2}{\|z\|^2} - \dfrac{\langle dz, z\rangle \odot \langle z, dz\rangle}{\|z\|^4}$

Taking a local section s over $U_0 \subset \mathbb{C}P^n$ (corresponding to $z^0 \neq 0$) of $q \circ p : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$, for example, we take $(z^1, \dots, z^n) \xrightarrow{s} (1, z^1, z^2, \dots, z^n)$, and pulling back $p^*(q^* g_{F-S})$ by s, we get:

$g_{F-S} = s^*\left(p^*\left(q^* g_{F-S}\right)\right)$

$$\boxed{g_{F-S} = \left(\frac{\sum_{i=1}^{n} dz^i \odot d\overline{z^i}}{1 + \sum_{i=1}^{n} |z^i|^2}\right) - \left(\frac{\sum_{i,j=1}^{n} \overline{z^i} z^j \, dz^i \odot d\overline{z^j}}{\left(1 + \sum_{i=1}^{n} |z^i|^2\right)^2}\right)}$$

over $U_0 \subset \mathbb{C}P^n$.

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2) a) $\{a_{11}a_{22} = 1$ and $a_{12} = a_{13} = a_{21} = a_{31} = a_{32} = 0\} \cap GL(3, \mathbb{R})$ is a closed subset of $GL(3,\mathbb{R})$ with 2 connected components, depending on whether $a_{22} > 0$ or $a_{22} < 0$, the first connected component being G. Thus $G \subseteq GL(3,\mathbb{R})$ is a closed subset of $GL(3,\mathbb{R})$.

(In fact $G = \{a_{11}a_{22} = 1, a_{12} = a_{13} = a_{21} = a_{31} = a_{32} = 0$ and $a_{22} \geq 0\} \cap GL(3,\mathbb{R})$)

It is also easy to see that G is a subgroup of $GL(3,\mathbb{R})$.

b) Let

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$g_1, g_2 \in G$ and $g_1 g_2 \neq g_2 g_1$, so G is non-abelian.

c) Proof by contradiction. Suppose that G admits a biinvariant pseudo-Riemannian metric, then there is an Ad_G-invariant pseudo-inner-product on $\mathfrak{g}$, which we denote by $(-,-)$. Then, for any $g \in G$, $(Ad_g(X), Ad_g(Y)) = (X, Y)$, from which it follows that $\det(Ad_g) = \pm 1$ for all $g \in G$.

We compute:

$$Ad_g(X) = \underbrace{\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}}_{g} \underbrace{\begin{pmatrix} -x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 0 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}^{-1}}_{g^{-1}} = \begin{pmatrix} -x & 0 & 0 \\ 0 & x & ay - bx \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad Ad_g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \det(Ad_g) = a$$

So if we take a $g \in G$ having $a \neq 1$, we get a contradiction.

3) a) (i) $SO(2,1)$

Let $h = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Then $SO(2,1) = \{ g \in GL(2;\mathbb{R}) \mid g^T h g = h \}$

Its Lie algebra $so(2,1) = \{ X \in gL(3;\mathbb{R}) \mid X^T h + hX = 0 \}$

$$\Rightarrow so(2,1) = \left\{ \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & -x & 0 \end{pmatrix} \in gL(3,\mathbb{R}) \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

Let $X_i = \begin{pmatrix} 0 & z_i & y_i \\ z_i & 0 & x_i \\ y_i & -x_i & 0 \end{pmatrix}$, $i = 1, 2$.

$$\Rightarrow [X_1, X_2] = \begin{pmatrix} 0 & x_1 y_2 - x_2 y_1 & z_1 x_2 - z_2 x_1 \\ x_1 y_2 - x_2 y_1 & 0 & z_1 y_2 - z_2 y_1 \\ x_2 z_1 - x_1 z_2 & y_1 z_2 - y_2 z_1 & 0 \end{pmatrix}$$

$$\Rightarrow ad_{X_1}(X_2) = \begin{pmatrix} 0 & z_1 & -y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\Rightarrow (X_1, X_2)_{killing} = tr\left(ad_{X_1} \circ ad_{X_2} \right)$$

$$\boxed{(X_1, X_2)_{killing} = 2(-x_1 x_2 + y_1 y_2 + z_1 z_2)}$$

ii) Nil = Heisenberg group

Its Lie algebra nil is $nil = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \in gL(3,\mathbb{R}) \;\middle|\; x,y,z \in \mathbb{R} \right\}$

Let $X_i = \begin{pmatrix} 0 & x_i & z_i \\ 0 & 0 & y_i \\ 0 & 0 & 0 \end{pmatrix}$, $i = 1, 2$

$$\Rightarrow [X_1, X_2] = \begin{pmatrix} 0 & 0 & x_1 y_2 - x_2 y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathrm{ad}_{X_1}(X_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -y_1 & x_1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\Rightarrow \boxed{(X_1, X_2)_{killing} = tr(\mathrm{ad}_{X_1} \circ \mathrm{ad}_{X_2}) = 0}$$

b) We compute the left-invariant Maurer-Cartan 1-form on Nil, $\theta_{M-C} \in C^\infty(G, T^*G \otimes g)$, defined by:

$\theta_{M-C}\big|_g = g^{-1} dg$, i.e.

$$\theta_{M-C} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & dx & dz \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & dx & dz - x\,dy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}$$

It follows that dx, dy and $dz - x\,dy$ are left-invariant
1-forms on Nil, which are pointwise linearly independent.
So an example of a left-invariant metric on Nil is

$$\boxed{g = dx^2 + dy^2 + (dz - x\,dy)^2}$$

c) $$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k & n \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+k & z+mx+n \\ 0 & 1 & y+m \\ 0 & 0 & 1 \end{pmatrix}$$

Define a map $F : \mathrm{Nil}/\Gamma \longrightarrow \mathbb{R}/\mathbb{Z}$,

which sends $$\left[\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right] \longmapsto [x] \in \mathbb{R}/\mathbb{Z}$$

F is smooth and onto, with fibres given by:

$F^{-1}([x]) = \mathbb{R}^2/\Lambda_{[x]}$, where $\Lambda_{[x]}$ is the 2-dimensional

lattice spanned by $\begin{pmatrix} 1 \\ x \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\Lambda_{[x]} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} y \\ z \end{pmatrix} = m\begin{pmatrix} 1 \\ x \end{pmatrix} + n\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ where } m, n \in \mathbb{Z} \right\}$$

<u>Remark</u>: $\Lambda_{[x]}$ does not get affected if x is replaced by $x+k$, $k \in \mathbb{Z}$.

Hence Nil/Γ is a T^2 fibration over S^1, and is thus compact.