Solution to Homework 7:

1) $(M, g) \to$ Riemannian manifold

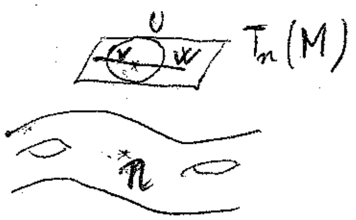
$\varphi : M \to M$ isometry

$N = \{ n \in M \mid \varphi(n) = n \}$.

a) if $\gamma(t)$ is a geodesic in M, then $\varphi(\gamma(t))$ is also a geodesic in M, because $\varphi^* \nabla = \nabla$, ∇ being the L-C connection, using the fact that φ is an isometry.

Let $n \in N$. Let $U \subseteq T_n(M)$ be an open ball centered at 0 such that $\exp_n|_U$ is a diffeomorphism. Let $W \subseteq T_n(M)$ be the eigenspace of $d\varphi_n$ for the eigenvalue 1.

Let $V = W \cap U$.



Then if $L \subset V$ is a line segment passing through $0 \in T_n(M)$, then $\exp_n(L)$ is a geodesic in M passing through n, with initial velocity fixed by $d\varphi_n$. So $\varphi(L)$ is also a geodesic passing through n with the same initial velocity. Hence, by uniqueness of geodesics, $\varphi(L) = L$.

$\Rightarrow \exp_n(V) \subseteq N$

Similarly, $\exp_n(U - V) \subseteq M - N$

This shows that (each connected component of) γN is a smooth submanifold of M.

b) Suppose that I is a connected interval.
We have seen that, if $S = \{ t \in I \mid \gamma(t) \in N \}$, then S is an open subset of I. We have also seen that $N \subseteq M$ is closed, so that $S = \gamma^{-1}(N)$ is closed in I. Hence, S is both open and closed in I, and I is connected, so either $S = \emptyset$ or $S = I$.
But $S \ni t_0$, so $S \neq \emptyset$, which implies that $S = I$.
Therefore, N is a totally geodesic submanifold of M.

(2) $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$

$$g = \frac{4\, dz \odot d\bar{z}}{(1 - |z|^2)^2}$$

if $z \mapsto \lambda z$, then
$$dz \mapsto \lambda\, dz$$
$$d\bar{z} \mapsto \bar{\lambda}\, d\bar{z}$$
so $dz \odot d\bar{z} \mapsto |\lambda|^2 dz \odot d\bar{z} = dz \odot d\bar{z}$

So the rotations $z \mapsto \lambda z$ $(\lambda \in U(1))$ are isometries.

if $z \overset{\varphi}{\mapsto} \bar{z}$, then
$$dz \mapsto d\bar{z}$$
$$d\bar{z} \mapsto dz$$
so $dz \odot d\bar{z} \mapsto d\bar{z} \odot dz = dz \odot d\bar{z}$

& $\varphi^*(g) = g$, so the reflection φ is an isometry.

The fixed point set of φ is $\mathbb{R} \cap D$, i.e. the real axis intersected with D, which is 1-dimensional totally geodesic submanifold of D (by problem 1) a) & b)), so $\mathbb{R} \cap D$ is a geodesic in the hyperbolic plane. Similarly, its image by any rotation $z \mapsto \lambda z$ is also a geodesic (since rotations are isometries).

b) $\psi(z) = \dfrac{z+\lambda}{1+\lambda z}$

$$\psi'(z) = \frac{1+\lambda z - \lambda(z+\lambda)}{(1+\lambda z)^2} = \frac{1-\lambda^2}{(1+\lambda z)^2}$$

$$\Rightarrow \boxed{\psi^*(dz) = \frac{(1-\lambda^2)}{(1+\lambda z)^2}\, dz} \quad \Rightarrow \boxed{\psi^*(d\bar z) = \frac{(1-\lambda^2)}{(1+\lambda \bar z)^2}\, d\bar z}$$

$$\& \boxed{\psi^*\left(\frac{4}{(1-|z|^2)^2}\right) = \frac{4}{(1-|\psi(z)|^2)^2}}$$

After a short computation, we get that $\boxed{\psi^*(g) = g}$ (ψ is an isometry).

I claim that $\Psi(it)$, for $-1 < t < 1$, lies on the circle $\mathcal{C}_\lambda$ centered at $\frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)$ & radius $\frac{1}{2}\left|\frac{1}{\lambda} - \lambda\right|$, if $\lambda \neq 0$, or on the line $i\mathbb{R} \cap D$ if $\lambda = 0$. This can be seen by checking that $\left|\Psi(it) - \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)\right|^2 = \frac{1}{4}\left(\frac{1}{\lambda} - \lambda\right)^2$.

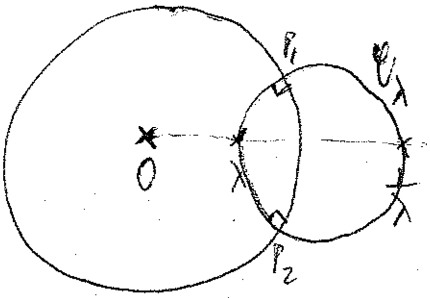
We still need to check that $\mathcal{C}_\lambda$ meets the unit circle orthogonally.

<u>Lemma:</u> if $\tau : \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ is the inversion $\tau(z) = \frac{z}{|z|^2} = \frac{1}{\bar{z}}$, then $\tau(\mathcal{C}_\lambda) = \mathcal{C}_\lambda$. It suffices to check that 3 distinct points on $\mathcal{C}_\lambda$ are mapped by τ to 3 distinct points on $\mathcal{C}_\lambda$: one can take say λ, $\frac{1}{\lambda}$ and one of the two points of intersection of $\mathcal{C}_\lambda$ with the unit circle.

Let p_1, p_2 be those 2 points of intersection of $\mathcal{C}_\lambda$ with unit circle. The tangent line to $\mathcal{C}_\lambda$ at p_1 is an eigenspace of $d\tau_{p_1}$ since $\mathcal{C}_\lambda$ is preserved by τ. But $d\tau_{p_1}$ has the following eigenspaces:

* eigenspace for eigenvalue 1, which is the tangent line to the unit circle at p_1
* eigenspace for eigenvalue -1, which is the normal line to the unit circle at p_1.

But $\mathscr{C}_\lambda$ is not tangent to the unit circle at p_1, so it has to be orthogonal to the unit circle at p_1, and similarly at p_2.



(3) Lift the initial point of the geodesics to a point $p \in S^{2n+1}$ via π. WLOG, say

$$p = (0, \dots, 0, i) \in \mathbb{C}^{n+1} - \{0\}$$

Each geodesic γ_i on $\mathbb{C}P^n$, lifts to a geodesic $\tilde{\gamma}_i$ passing through p and initial velocity vector $\tilde{v}_i$ which is orthogonal to X (X being the vector field on S^{2n+1} generated by the $U(1)$ action on S^{2n+1}).

Thus, for some $\tilde{q}_i \in \mathbb{C}^n$, $\|\tilde{q}_i\| = 1$, we have

$$\tilde{\gamma}_i(t) = (\sin(t)\, \tilde{q}_i \,,\, i\cos(t))$$

<u>Case 1:</u> $\tilde{q}_1, \tilde{q}_2$ are lin. indep. over $\mathbb{C}$.

Then $\sin(T) = 0 \Rightarrow \boxed{T = \pi}$

<u>Case 2:</u> $\tilde{q}_2 = \lambda \tilde{q}_1$, for some $\lambda \in U(1)$.

Then $\sin(T)\cos(T) = 0 \Rightarrow \sin(2T) = 0 \Rightarrow \boxed{T = \frac{\pi}{2}}$.

Optional Problem A:

Since $\nabla_X Y = \frac{1}{2}[X,Y]$ is left-invariant if X & Y are left-invariant, it follows that ∇ is left-invariant.

Let Z be a left-invariant vector field, then:

$$[Z, \nabla_X Y] = \frac{1}{2}[Z,[X,Y]] = \frac{1}{2}[[Z,X],Y] + \frac{1}{2}[X,[Z,Y]]$$
$$= \nabla_{[Z,X]} Y + \nabla_X [Z,Y]$$

So ∇ is adjoint invariant, and therefore ∇ is biinvariant.

$\frac{1}{2}[dj(X), dj(Y)] = \frac{1}{2} dj[X,Y]$, so ∇ is j-invariant.

Moreover, $T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$ is X, Y are left-invariant vector fields. $\Rightarrow T^\nabla \equiv 0$ (since the left-invariant vector fields span the tangent space ~~at $g \in G$~~ $T_g(G)$ for any $g \in G$).

So ∇ is a biinvariant, j-invariant, torsion free linear connection on G.

Conversely, let ∇' be a biinvariant, j-invariant, torsion-free linear connection on G, and let,

$\nabla'_X Y = \nabla_X Y + A(X,Y)$, where $A \in C^\infty(G, T^* \otimes T^* \otimes T)$

But ∇' and ∇ are both left-invariant.

So $A \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$, by identifying $\mathfrak{g} \simeq \{$left-invariant vector fields on $\mathfrak{g}\}$.

Also, $(T^{\nabla'} - T^{\nabla})(X,Y) = A(X,Y) - A(Y,X)$

But $T^{\nabla'} = T^{\nabla} = 0$, so $A(X,Y) = A(Y,X)$ for any left-invariant vector fields.

Finally, let e^{tx} be a 1-parameter subgroup of G, with $x \in \mathfrak{g}$. ~~Then, if X is the left-invariant vect. field corresponding to $x \in \mathfrak{g}$,~~

$\cancel{\left(\nabla'_{\frac{d}{dt}}\right)^2 e^{tx} = \nabla'_X X}$ and X is the left-invariant vect. field corresponding to $x \in \mathfrak{g}$.

$$\nabla'_{dj(X)} dj(X) = dj(\nabla'_X X) \quad \text{since } \nabla' \text{ is } j\text{-invariant},$$

so $\quad \nabla'_X X = -\nabla'_X X$

$\Rightarrow \nabla'_X X = 0 \quad \forall\ X$ left-invariant vect. field.

$\Rightarrow A(X,Y) = -A(Y,X)$, i.e. A is skew-symmetric.

So A is both symmetric and skew-symmetric, therefore $A = 0$, and $\nabla' = \nabla$, proving uniqueness.