1) $\tilde{D}_{\tilde{X}}\tilde{Y} = (\tilde{D}_{\tilde{X}}\tilde{Y})^h + (\tilde{D}_{\tilde{X}}\tilde{Y})^v$

$$2\tilde{g}(\tilde{D}_{\tilde{X}}\tilde{Y}, \tilde{Z}) = \tilde{X}\tilde{g}(\tilde{Y},\tilde{Z}) + \tilde{Y}\tilde{g}(\tilde{X},\tilde{Z}) - \tilde{Z}\tilde{g}(\tilde{X},\tilde{Y})$$
$$+ \tilde{g}([\tilde{X},\tilde{Y}],\tilde{Z}) - \tilde{g}([\tilde{X},\tilde{Z}],\tilde{Y}) - \tilde{g}([\tilde{Y},\tilde{Z}],\tilde{X})$$
$$= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)$$
$$+ g([X,Y],Z) - \ldots$$
$$= 2g(D_XY, Z) = 2\tilde{g}(\widetilde{D_XY}, \tilde{Z})$$

$$\Rightarrow \boxed{(\tilde{D}_{\tilde{X}}\tilde{Y})^h = \widetilde{(D_XY)}}$$

$$2\tilde{g}(\tilde{D}_{\tilde{X}}\tilde{Y}, U) = \tilde{g}([\tilde{X},\tilde{Y}], U)$$

$$\Rightarrow (\tilde{D}_{\tilde{X}}\tilde{Y})^v = \frac{1}{2}[\tilde{X},\tilde{Y}]^v$$

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a) $\tilde{g}(\tilde{D}_U\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{D}_{\tilde{X}}U, \tilde{Y}) = -\tilde{g}(\tilde{D}_{\tilde{X}}\tilde{Y}, U)$

$$= -\frac{1}{2}\tilde{g}([\tilde{X},\tilde{Y}]^v, U)$$

$$\tilde{g}(\tilde{W}, \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}) = \tilde{g}(\tilde{D}_{\tilde{X}}\tilde{D}_{\tilde{Y}}\tilde{Z} - \tilde{D}_{\tilde{Y}}\tilde{D}_{\tilde{X}}\tilde{Z} - \tilde{D}_{[\tilde{X},\tilde{Y}]}\tilde{Z}, \tilde{W})$$

$$\tilde{g}(\tilde{W}, \tilde{D}_{\tilde{X}}\tilde{D}_{\tilde{Y}}\tilde{Z}) = \tilde{g}(\tilde{D}_{\tilde{X}}(\widetilde{D_YZ} + \frac{1}{2}[\tilde{Y},\tilde{Z}]^v), \tilde{W})$$
$$= \tilde{g}(\widetilde{D_XD_YZ} + \frac{1}{2}[\tilde{X},\widetilde{D_YZ}]^v$$
$$= \tilde{g}(\tilde{W}, \widetilde{D_XD_YZ} + \frac{1}{2}[\tilde{X},\widetilde{D_YZ}]^v)$$
$$- \frac{1}{4}\tilde{g}([\tilde{Y},\tilde{Z}]^v, [\tilde{X},\tilde{W}]^v)$$

$$-\tilde{g}(\tilde{D}_{[\hat{X},\hat{Y}]}\tilde{Z},\tilde{W}) = -\tilde{g}(\tilde{D}_{\widetilde{[X,Y]}}\tilde{Z},\tilde{W}) - \tilde{g}(\tilde{D}_{[\hat{X},\hat{Y}]^v}\tilde{Z},\tilde{W})$$

$$= -\hat{g}(\widetilde{D_{[X,Y]}Z},\tilde{W}) - \frac{1}{2}\tilde{g}([\widetilde{[X,Y]},\tilde{Z}]^v,\tilde{W})$$

$$+ \frac{1}{2}\tilde{g}([\hat{X},\hat{Y}]^v,[\tilde{Z},\tilde{W}]^v)$$

$$\Rightarrow \boxed{\begin{aligned}\tilde{R}(\hat{X},\hat{Y},\hat{Z},\hat{W}) = R(X,Y,Z,W) &- \frac{1}{4}\tilde{g}([\hat{Y},\hat{Z}]^v,[\hat{X},\hat{W}]^v) \\ &+ \frac{1}{4}\tilde{g}([\tilde{X},\tilde{Z}]^v,[\tilde{Y},\tilde{W}]^v) + \frac{1}{2}\tilde{g}([\hat{X},\hat{Y}]^v,[\tilde{Z},\tilde{W}]^v)\end{aligned}}$$

$$\Rightarrow \tilde{K}(\hat{X},\hat{Y}) = K(X,Y) - \frac{3}{4}\tilde{g}([\hat{X},\hat{Y}]^v,[\hat{X},\hat{Y}]^v)$$

$$\boxed{\tilde{K}(\hat{X},\hat{Y}) = K(X,Y) - \frac{3}{4}\|[\hat{X},\hat{Y}]^v\|^2}$$

c) By O'Neill's formula, it suffices to show that on G, with biinvariant metric, if $X,Y \in \mathfrak{g}$ are unit length and orthogonal, then:

$$K(X,Y) = \frac{1}{4}\|[X,Y]\|^2$$

$$D_X D_Y Z = \frac{1}{2}D_X([Y,Z]) = \frac{1}{2}[D_X Y, Z] + \frac{1}{2}[Y, D_X Z]$$

$$- D_Y D_X Z = -\frac{1}{2}[D_Y X, Z] - \frac{1}{2}[X, D_Y Z]$$

$$- D_{[X,Y]}Z = -D_Z([X,Y]) - [[X,Y],Z] = -[D_Z X, Y] - [X, D_Z Y] - [[X,Y],Z]$$

$$\Rightarrow \cancel{R(X,Y,Z,W)} =$$

$$
\begin{aligned}
R(X,Y,Z,W) &= \tfrac{1}{2}\left(\cancel{[D_X Y, Z]}, W\right) + \tfrac{1}{2}\left(\cancel{[Y, D_X Z], W}\right) \\
&\quad - \tfrac{1}{2}\left(\cancel{[D_Y X, Z]}, W\right) - \tfrac{1}{2}\left(\cancel{[X, D_Y Z]}, W\right) \\
&\quad - \left(\cancel{[D_Z X, Y]}, W\right) - \left(\cancel{[X, D_Z Y]}, W\right) \\
&\quad - \left(\cancel{[[X,Y],Z]}, W\right) \\
&= -\tfrac{1}{2}\left([[X,Y],Z], W\right) - \tfrac{1}{2}\left([X, D_Y Z], W\right) \\
&\quad - \left([X, D_Z Y], W\right) + \tfrac{1}{2}\left([Y, D_X Z], W\right) - \left([D_Z X, Y], W\right)
\end{aligned}
$$

$$
\boxed{R = -\tfrac{1}{2}\left([[X,Y],Z], W\right) + \tfrac{1}{4}\left([X,[Y,Z]], W\right) - \tfrac{1}{4}\left([Y,[X,Z]], W\right)}
$$

$$
\begin{aligned}
\Rightarrow R(X,Y,Y,X) &= -\tfrac{1}{2}\left([[X,Y],Y], \cancel{X}\right) - \tfrac{1}{4}\left([Y,[X,Y]], \cancel{X}\right) \\
&= +\tfrac{1}{4}\left([Y,[X,Y]], X\right)
\end{aligned}
$$

$$
\boxed{k(X,Y) = \tfrac{1}{4}\,\|[X,Y]\|^2}
$$

$\mathbb{CP}^2$:

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{x} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{y} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow [x, y] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow [x, y]$$

$$\Rightarrow \underline{k(x, y) = 1}$$

$$\boxed{k(X, Y) = \frac{1}{4} \| [X, Y]^p \|^2 + \| [X, Y]^h \|^2}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{x'} \underbrace{\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}}_{y'} = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$[x', y'] = 2 \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$\Rightarrow \underline{k(x', y') = 4}$$

$$w = \cos(\alpha)\, w_0 + \sin(\alpha)(iv)$$

$[v, w_0] \in \mathfrak{h}$

$\|[v, w_0]\|^2 = 1$

$[v, (iv)] \in \mathfrak{h}$

$\|[v, (iv)]\|^2 = 4$

$\Rightarrow [v, w] = \cos(\alpha)[v, w_0] + \sin(\alpha)[v, iv] \in \mathfrak{h}$

$\Rightarrow \|[v, w]\|^2 = \cos^2(\alpha) + 4\sin^2(\alpha)$

Also $[v, w_0] \perp [v, iv]$

$$\boxed{k(v, w) = 1 + 3\sin^2(\alpha)}$$