Solution to Homework 9:

1 a) $\tilde{D}_U V = D_U V + (\tilde{D}_U V)^{\perp}$

$\Rightarrow (\tilde{D}_U V)^{\perp} = \tilde{D}_U V - D_U V$

$(\tilde{D}_{fU} V)^{\perp} = f(\tilde{D}_U V)^{\perp} \quad \forall f \in C^{\infty}(M)$

$(\tilde{D}_U (fV))^{\perp} = U(f)V - U(f)V + f\tilde{D}_U V - f D_U V$

$= f(\tilde{D}_U V)^{\perp}$

$(\tilde{D}_V U)^{\perp} = \tilde{D}_V U - D_V U$

$= \tilde{D}_U V - D_U V + [V,U] + [U,V]$

$= (\tilde{D}_U V)^{\perp}$

$\Rightarrow \boxed{\ell(v,u) = \ell(u,v)}$

b) $\tilde{R}(X,Y,U,V) = g(-\tilde{D}_X \tilde{D}_Y U + \tilde{D}_Y \tilde{D}_X U + \tilde{D}_{[X,Y]} U, V)$

$\tilde{D}_Y U = D_Y U + \ell(Y,U)\nu$

$-\tilde{D}_X \tilde{D}_Y U = -D_X D_Y U - \ell(X, D_Y U)\nu - X(\ell(Y,U))\nu$

$- \ell(Y,U)\tilde{D}_X \nu$

$\tilde{D}_{[X,Y]} U = D_{[X,Y]} U + \ell([X,Y],U)\nu$

$\Rightarrow \boxed{\tilde{R}(X,Y,U,V) = R(X,Y,U,V) - \ell(Y,U)g(\tilde{D}_X \nu, V) + \ell(X,U)g(\tilde{D}_Y \nu, V)}$

But $g(\tilde{D}_X \nu, V) = -g(\tilde{D}_X V, \nu) = -\ell(X,V)$

and $g(\tilde{D}_Y \nu, V) = -\ell(Y,V)$, so we proved the formula.

2

c) $\ell(u,v) = g\left(\tilde{\nabla}_U V, \frac{\tilde{\nabla} f}{\|\tilde{\nabla} f\|}\right)$

$$= -\frac{1}{\|\tilde{\nabla} f\|}\, g(V, \tilde{\nabla}_U \tilde{\nabla} f)$$

$$= -\frac{1}{\|\tilde{\nabla} f\|}\, g(V, \tilde{\nabla}(\tilde{\nabla}_U f)) \quad (\text{since } \tilde{\nabla} \to \text{torsion-free})$$

$$= -\frac{1}{\|\tilde{\nabla} f\|}\, (\tilde{\nabla}_V \tilde{\nabla}_U f)$$

$$\boxed{\ell(u,v) = -\frac{1}{\|\tilde{\nabla} f\|}\, (\mathrm{Hess}\, f)(U, V)}$$

c) if X, Y are two lin. indep. vect. fields, not necessarily ~~linearly indep.~~ orthonormal, then it follows from b) that:

$$\tilde{K}(X,Y) = K(X,Y) + \frac{\ell(X,Y)^2 - \ell(X,X)\,\ell(Y,Y)}{\|X \wedge Y\|^2}$$

But $\tilde{K} \equiv 0$, so $\boxed{K(X,Y) = \frac{\ell(X,X)\,\ell(Y,Y) - \ell(X,Y)^2}{\|X \wedge Y\|^2}}$

$$f = x^2 + y^2 - z^2 + 1 \quad \Rightarrow \quad \nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix}, \quad \|\nabla f\| = 2r,$$

$$\nu = \frac{1}{r}\begin{pmatrix} x \\ y \\ -z \end{pmatrix}, \quad \mathrm{Hess}\, f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

we take $X = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ z \\ y \end{pmatrix}$

$$\Rightarrow \ l(X,X) = \frac{1}{2r}\begin{pmatrix} z & 0 & x \end{pmatrix}\begin{pmatrix} 2 & & \\ & 2 & \\ & & -2 \end{pmatrix}\begin{pmatrix} z \\ 0 \\ x \end{pmatrix} = -\left(\frac{z^2 - x^2}{r}\right)$$

similarly, $l(X,Y) = \frac{xy}{r}$

$$l(Y,Y) = -\frac{(z^2 - y^2)}{r}$$

$$\|X \wedge Y\|^2 = z^2 r^2$$

$$\Rightarrow \ K = \frac{(z^2 - x^2)(z^2 - y^2) - x^2 y^2}{z^2 r^4}$$

$$= \frac{1}{r^4}\left(z^2 - x^2 - y^2\right)$$

$$\boxed{K = \frac{1}{r^4}} \quad \text{since } z^2 - x^2 - y^2 = 1$$

(here $r^2 = x^2 + y^2 + z^2$)