An Introduction to Twistor Theory

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Abstract

We provide a very short introduction to twistor theory. Twistor theory establishes a one-to-one correspondence between anti-self-dual 4-manifolds on one hand, and complex 3-folds containing a rational curve with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ and endowed with a real structure, on the other hand.

1 Basic Example: $\mathbb{C}P^3$ and $\mathbb{H}P^1 \simeq S^4$

Define the map

$$p : \mathbb{C}P^3 \to \mathbb{H}P^1$$

sending a point $z \in \mathbb{C}P^3$, which corresponds to a line $L_z \subset \mathbb{C}^4$, to its quaternionification $\mathbb{H}L_z \in \mathbb{H}P^1$, by identifying $\mathbb{C}^4 \simeq \mathbb{H}^2$. The complex 3-fold $\mathbb{C}P^3$ is endowed with a real structure σ; in other words, $\sigma : \mathbb{C}P^3 \to \mathbb{C}P^3$ is an antiholomorphic involution on $\mathbb{C}P^3$. The real structure σ is induced by the quaternionic structure $j : \mathbb{C}^4 \to \mathbb{C}^4$, defined by

$$\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \stackrel{j}{\longmapsto} \begin{pmatrix} -\overline{z}_1 \\ \overline{z}_0 \\ -\overline{z}_3 \\ \overline{z}_2 \end{pmatrix}.$$

Given $x \in \mathbb{H}P^1$, $p^{-1}(x)$ is a holomorphically embedded $\mathbb{C}P^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ in $\mathbb{C}P^3$, which is in addition preserved by σ. A holomorphically embedded $\mathbb{C}P^1$ in $\mathbb{C}P^3$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is called a twistor line. A subset of a complex manifold endowed with a real structure is said to be real if it is preserved by the real structure. Hence the fibers $p^{-1}(x)$ of p are real twistor lines, and they are in fact *all* real twistor lines in $\mathbb{C}P^3$.

More generally, a twistor line is just a projective line in $\mathbb{C}P^3$, and the manifold of all twistor lines in $\mathbb{C}P^3$ is the complex grassmannian

$$Gr_2(\mathbb{C}^4) = SU(4)/S(U(2) \times U(2)).$$

The real structure σ on $\mathbb{C}P^3$ induces a real structure τ on $Gr_2(\mathbb{C}^4)$, and $\mathbb{H}P^1$ is the submanifold of real points in $Gr_2(\mathbb{C}^4)$ via the identification $\mathbb{C}^4 \simeq \mathbb{H}^2$. From the Plcker embedding, it follows that

$$Gr_2(\mathbb{C}^4) \simeq Q^4,$$

where Q^4 is a smooth non-degenerate quadric in $\mathbb{C}P^5$.

One defines F to be the partial variety $F_{1,2}(\mathbb{C}^4)$, consisting of pairs (L, P), where L (respectively P) is a one-dimensional (respectively two-dimensional) subspace of $\mathbb{C}^4$, and $L \subset P$. The partial flag variety F is a complex 5-fold, and we have the following double fibration:

$$\begin{array}{ccc} & F & \\ {\scriptstyle \mu}\swarrow & & \searrow{\scriptstyle \nu} \\ \mathbb{C}P^3 & & Gr_2(\mathbb{C}^4) \end{array}$$

The map μ maps (L, P) to L, while the map ν maps (L, P) to P. This double fibration plays a fundamental role in the Penrose transform. Roughly speaking, starting from a holomorphic "field" on $\mathbb{C}P^3$, you first pullback that field to F via μ, then integrate it along the fibers of ν to obtain a field on $Gr_2(\mathbb{C}^4)$ satisfying some linear differential equation (for example, the Dirac and Laplace equations).

One calls $\mathbb{C}P^3$ the twistor space of $\mathbb{H}P^1$ (historically, Penrose called it the projective twistor space, and called twistor a solution to the twistor equation, which is in some sense an orthogonal complement of the Dirac equation).

I will write more when I have more free time.