Differential Forms and Applications

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Abstract

In this talk, we introduce the language of differential forms and the exterior differential d by restricting to $\mathbb{R}^n$. We state Stoke's theorem in this setting, and explain how it includes Green's theorem as well as Gauss's theorem of multivariable calculus as special cases. As applications, we give a very short proof of Cauchy's theorem in complex analysis, as well as a very well known way of writing Maxwell's equations using differential forms, the d-operator as well as the Hodge star. The material is by now very well known, but the hope is to demonstrate the beauty and usefulness of differential forms.

1 Definitions and Basic Properties

Let V be a real vector space of dimension n, and chose a basis $e_1, \ldots, e_n$ for V.

We would like to define $\Lambda^k V$, the k-th exterior power of V. First, if S is any set, then we define $F(S)$ to be the free vector space generated by S. More precisely,

$$F(S) = \{c_1.s_1 + \ldots + c_l.s_l; c_i \in \mathbb{R} \text{ and } s_i \in S\}.$$

We let I be the vector subspace of $F(V \times \ldots \times V)$ (the cartesian product of k copies of V) generated by elements of the form:

$$(v_1, \ldots, a.v_i + b.w_i, \ldots, v_k) - a.(v_1, \ldots, v_i, \ldots, v_k) - b.(v_1, \ldots, w_i, \ldots, v_k),$$
$$(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) + (v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k),$$

for all $a, b \in \mathbb{R}$, $v_j \in V$, $1 \leq j \leq n$, $w_i \in V$, and $1 \leq i \leq n$. We write $v_1 \wedge \ldots \wedge v_k$ for $[(v_1, \ldots, v_k)]$. Then, one can show that $\Lambda^k V$ is a vector space of dimension $\binom{n}{k}$. Given a basis $e_1, \ldots, e_n$ of V, one can show that the following form a basis of $\Lambda^k V$:

$$(e_{i_1} \wedge \ldots \wedge e_{i_k}; 1 \leq i_1 < \ldots < i_k \leq n).$$

Assume now that (V, g) is an inner product space: more precisely, $g : V \times V \to \mathbb{R}$ is a symmetric, bilinear and non-degenerate. The latter condition means that if there is a $v \in V$ such that $g(v, w) = 0$ for all $w \in V$, then $v = 0$. Examples include:

1. Euclidean n-dimensional space, which is $\mathbb{R}^n$ with the "Euclidean" inner product
$$g(v, w) = \sum_{i=1}^{n} v_i w_i.$$

2. Minkowski n-dimensional space, which is $\mathbb{R}^n$ with coordinates x_0, $\ldots$, x_{n-1} and the Lorentzian inner product
$$g(v, w) = -v_0 w_0 + \sum_{i=1}^{n-1} v_i w_i.$$

We next move on from multilinear algebra, to multivariable calculus. Consider $\mathbb{R}^n$ with coordinates x_i. A k-form α on $\mathbb{R}^n$ is a smooth map from $\mathbb{R}^n$ to $\Lambda^k(\mathbb{R}^n)^*$. It can be written as

$$\alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \alpha_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \quad = \frac{1}{k!} \sum_{i_1, \ldots, i_k} \alpha_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

where each $\alpha_{i_1 \ldots i_k}$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$ and, in the second line, we have extended $\alpha_{i_1 \ldots i_k}$ by complete skew-symmetry in its indices.

We now define the exterior differential, Cartan's famous d operator, which maps k-forms to $k+1$-forms, i.e. it increases the degree of a form by 1. We start off by defining df, where f is a 0-form, i.e. a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$.

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

We next wish to define d of a k-form. We do this in two steps.

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

We then extend this definition to arbitrary k-forms by $\mathbb{R}$-linearity. We compute some examples in $\mathbb{R}^3$:

1. We first remark that df is nothing but the gradient of f:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3.$$

2. We compute the d of a 1-form:

$$\begin{aligned} & d(f_1 dx_1 + f_2 dx_2 + f_3 dx_3) \\ & = df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\ & = (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}) dx_1 \wedge dx_2 + (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}) dx_2 \wedge dx_3 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}) dx_3 \wedge dx_1. \end{aligned}$$

   This is nothing but the curl of a vector field in $\mathbb{R}^3$.

3. We now compute the d of a 2-form:

$$\begin{aligned} & d(f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy) \\ & = (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}) dx \wedge dy \wedge dz. \end{aligned}$$

   This is nothing but the divergence of a vector field.

2 Stoke's Theorem

Let $B \subset \mathbb{R}^n$ be a $k+1$-dimensional compact oriented submanifold of $\mathbb{R}^n$ with boundary ∂B (∂B has dimension k). Let α be a k-form on $\mathbb{R}^n$. Then Stoke's theorem asserts that

$$\int_B d\alpha = \int_{\partial B} \alpha.$$

Taking α to be a 1-form in $\mathbb{R}^3$ and B to be an oriented surface in $\mathbb{R}^3$ with boundary the closed curve ∂B, we recover the classical Green's theorem.

On the other hand, taking α to be a 2-form in $\mathbb{R}^3$ and taking B to be a smooth 3-dimensional region in $\mathbb{R}^3$ with a closed surface ∂B as boundary, we recover Gauss's theorem.

3 Applications

Our first theorem is an application of Stoke's theorem to prove Cauchy's theorem in complex analysis. First, observe that for an arbitrary smooth complex-valued function $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$ is a non-empty open set, we get

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z},$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) \text{ and } \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

Hence, if f is holomorphic, we get

$$d(fdz) = df \wedge dz = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = 0.$$

Taking now B to be a star-shaped region in $\mathbb{C}$ with boundary $\Gamma = \partial B$, we obtain

$$\int_\Gamma f dz = \int_B d(fdz) = 0,$$

by Stoke's theorem, assuming f is holomorphic on an open set containing B. This proves Cauchy's theorem.

Our second application is to electromagnetism. Let E and B be the electric and magnetic fields on Minkowski four-dimensional spacetime, with B thought of as a 2-form and E as a 1-form. More precisely,

$$B = B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy \text{ and } E = E_1 dx + E_2 dy + E_3 dz.$$

Using E and B, one can form the 2-form

$$F = B + cdt \wedge E.$$

Also, define

$$j = -\rho_0 dx \wedge dy \wedge dz + J_1 dt \wedge dy \wedge dz + J_2 dt \wedge dz \wedge dx + J_3 dt \wedge dx \wedge dy,$$

with ρ_0 being the charge density and J being the current density.

With respect to F and j, Maxwell's equations can now be written as

$$dF = 0$$
$$d * F = 4\pi j,$$

with $*$ being the Hodge star, which maps a k-form to an $n-k$ form, and satisfies

$$\alpha \wedge *\beta = (\alpha, \beta) v_g,$$

where α and β are arbitrary k-forms, $(-,-)$ denotes the extension of the Minkowski inner product to k-forms and v_g is the volume form associated to the Minkowski inner product, i.e.

$$v_g = cdt \wedge dx \wedge dy \wedge dz.$$

We also remark that we are now using a slightly different definition than the previous definition of the Minkowski inner product, in order to incorporate the speed of light constant c.

$$(x, y) = -c^2 x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

A straight-forward computation shows that

$$dF = 0 \Leftrightarrow \begin{cases} \nabla . B = 0 \\ c\nabla \times E + \partial_t B = 0 \end{cases}$$

and

$$d * F = 4\pi j \Leftrightarrow \begin{cases} \nabla \times B - \frac{1}{c}\partial_t E = \frac{4\pi J}{c} \\ \nabla . E = 4\pi \rho_0 \end{cases}$$